

Chapter 5

Series Solutions of Second Order Linear Equations

In this chapter we consider solving second order linear equations with variable coefficients. To find two fundamental solutions it is necessarily to search for solution in a larger set of functions than the set of elementary functions. Power series representation of functions is our principal tool. The basic idea is to look for a solution in the form of a power series and try to find the coefficients in this series.

5.1 Review of Power Series

In this section we summarize very briefly the main results about power series that we need.

Recall that a power series is an infinite sum of the form

$$\sum_{n=0}^{\infty} a_n(x - x_0)^n = 0.$$

The constant x_0 is the center and the constants $a_0, a_1, \dots$ are the coefficients.

- (1) A power series $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ converges at a point x if $\lim_{M \rightarrow \infty} \sum_{n=0}^M a_n(x - x_0)^n$ exists.
- (2) A power series $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ certainly converges at $x = x_0$. The set of all points at which a power series converges is called the interval of convergence.
- (3) Let $L = \lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|}$. Then $\rho = \frac{1}{L}$ is called the radius of convergence and the power series $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ converges in the interval $|x - x_0| < \rho = \frac{1}{L}$.
- (4) If $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ and $\sum_{n=0}^{\infty} b_n(x - x_0)^n$ are two power series, then

$$\sum_{n=0}^{\infty} a_n(x - x_0)^n + \sum_{n=0}^{\infty} b_n(x - x_0)^n = \sum_{n=0}^{\infty} (a_n + b_n)(x - x_0)^n.$$

(5) If $f(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n$, $|x - x_0| < \rho$, then

$$f'(x) = \sum_{n=1}^{\infty} n a_n(x - x_0)^{n-1}, \quad |x - x_0| < \rho.$$

(6) If $\sum_{n=0}^{\infty} a_n(x - x_0)^n = \sum_{n=0}^{\infty} b_n(x - x_0)^n$, then $a_n = b_n$ for $n = 0, 1, 2, \dots$. In particular, if $\sum_{n=0}^{\infty} a_n(x - x_0)^n = 0$, then $a_n = 0$ for $n = 0, 1, 2, \dots$.

(7) As long as we preserve the order of the terms of a series, we can change the index of the series without altering its convergence. Just as we make changes of the variable of integration in a definite integral, we can change the indices of a series. For example

$$\sum_{n=0}^{\infty} a_n(x - x_0)^n = \sum_{m=1}^{\infty} a_{m-1}(x - x_0)^{m-1}.$$

5.2 Series Solutions near an Ordinary Point, Part I

In this section we consider how we can find a solution of the second order linear equation

$$P(x)y'' + Q(x)y' + R(x)y = g(x). \quad (1)$$

It is sufficient to consider the homogeneous equation

$$P(x)y'' + Q(x)y' + R(x)y = 0, \quad (2)$$

since the procedure for the corresponding nonhomogeneous equation is similar.

Many problems in mathematical physics leads to equations of the form (2) having polynomial coefficients; for example, the Bessel equation

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0, \quad (3)$$

where ν is a constant, and the Legendre equation

$$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0, \quad (4)$$

where α is a constant. For this reason, as well as to simplify the algebraic computations, we primarily consider the case in which the functions P , Q , and R are polynomials. However, as we will see, the method of solution is also applicable when P , Q , and R are general analytic functions.

Definition. (Ordinary Points)

Let $P(x)$, $Q(x)$, and $R(x)$ be polynomials with no common factors. A point x_0 is called an ordinary point of the equation

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

if $P(x_0) \neq 0$. On the other hand, if $P(x_0) = 0$, then x_0 is called a singular point.

Example 1. Determine the singular points of the equation

$$(x^2 - 2x - 3)y'' + 2xy' + 3(x + 2)y = 0.$$

Solution.

Example 2. Find the singular points of the equation

$$y'' - \frac{1}{x(x^2 + 1)}y' + xy = 0.$$

Solution.

Series Solutions near an Ordinary Point

Let x_0 be an ordinary point of $P(x)y'' + Q(x)y' + R(x)y = 0$. Since $P(x)$ is continuous, there is an interval I containing x_0 in which $P(x)$ is never zero. In that interval we can divide by $P(x)$ and write equation (2) as

$$y'' + p(x)y' + q(x)y = 0, \quad (5)$$

where $p(x) = Q(x)/P(x)$ and $q(x) = R(x)/P(x)$ are continuous functions. Hence, according to the existence and uniqueness Theorem, there exists in that interval a unique solution of equation (2) that

also satisfies the initial conditions $y(x_0) = y_0$, $y'(x_0) = y'_0$ for arbitrary values of y_0 and y'_0 . In this and the following section we discuss the solution of equation (2) in the neighborhood of an ordinary point.

We look for solutions of the form

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots \quad (6)$$

We assume that the series converges in the interval $|x - x_0| < \rho$ for some $\rho > 0$. Our aim is to determine the coefficients a_n , $n = 0, 1, 2, \dots$. This can be achieved by substituting the series and its derivative for y and its derivatives.

We illustrate the method by the following examples. The operations, such as differentiation, that are involved in the procedure are justified so long as we stay within the interval of convergence.

Example 1. Find a series solution of the equation

$$y'' + xy' + y = 0$$

about the point $x_0 = 0$.

Solution.

Example 2. Find a series solution of the equation

$$y'' + xy' + (x^2 + 2)y = 0$$

about the point $x_0 = 0$.

Solution.

Example 3. Solve the initial value problem

$$y'' - xy' - y = 0, \quad y(0) = 2, \quad y'(0) = 1.$$

Solution.

Example 4. Solve the equation $y'' - xy' - y = 0$ by means of a power solution about $x = 1$.

Solution.

Example 5. Solve the initial value problem

$$(x^2 + 4)y'' + 6xy' + 4y = 0, \quad y(0) = 1, \quad y'(0) = 1.$$

Solution.

Example 6. Find a series solution of the equation

$$y'' - x^3y' - x^2y = 0$$

about the point $x_0 = 0$.

Solution.

5.3 Series Solutions near an Ordinary Point, Part II

In Section 5.2 we studied a method to find solutions of

$$P(x)y'' + Q(x)y' + R(x)y = 0, \quad (7)$$

where P , Q , and R are polynomials, in the neighborhood of an ordinary point x_0 . Assuming that the equation has a solution $y = \phi(x)$ that has a Taylor series

$$y = \phi(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n,$$

which converges for $|x - x_0| < \rho$, for some $\rho > 0$, we found that the coefficients a_n can be determined by directly substituting the series for y in the equation.

In this section we will consider how we might justify the statement that if x_0 is an ordinary point of equation (7), then there exist solutions of the form

$$y = \phi(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n.$$

In addition we consider the question of radius of convergence of such series. In doing this we are led to a generalization of the definition of an ordinary point.

Starting from the assumption that

$$y = \phi(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n$$

is a solution of equation (7), then by differentiating m times and sitting $x = x_0$ we find

$$m!a_m = \phi^{(m)}(x_0).$$

Thus, in order to compute a_n we must determine $\phi^{(n)}(x_0)$ for $n = 0, 1, 2, \dots$ from the differential equation.

Suppose that $y = \phi(x)$ satisfies the initial conditions $y(x_0) = y_0$, $y'(x_0) = y'_0$. Then $a_0 = y_0$ and $a_1 = y'_0$. If we are solely interested in finding a solution without specifying and initial conditions, then a_0 and a_1 remain arbitrary. In order to determine $\phi^{(n)}(x_0)$ and the corresponding a_n for $n = 2, 3, \dots$, we use the equation

$$P(x)y'' + Q(x)y' + R(x)y = 0.$$

Since ϕ is a solution, we have

$$P(x)\phi''(x) + Q(x)\phi'(x) + R(x)\phi(x) = 0.$$

For the interval about x_0 for which $P(x) \neq 0$ we can write this equation in the form

$$\phi''(x) = -p(x)\phi'(x) - q(x)\phi(x), \quad (8)$$

where $p(x) = \frac{Q(x)}{P(x)}$, $q(x) = \frac{R(x)}{P(x)}$. Setting $x = x_0$ in the above equation gives

$$\phi''(x_0) = -p(x_0)\phi'(x_0) - q(x_0)\phi(x_0).$$

Hence we can find $a_2 = \frac{1}{2!}\phi''(x_0)$.

To determine a_3 we differentiate equation (8) and then we set $x = x_0$ obtaining

$$a_3 = \frac{1}{3!}\phi'''(x_0) = -\frac{1}{3!}\left[p\phi'' + (p' + q)\phi' + q'\phi\right]_{x=x_0}.$$

Since P , Q , and R are polynomials and $P(x_0) \neq 0$, all of the derivatives of p and q exist at x_0 . Hence we can continue to differentiate equation (8) indefinitely, determining after each differentiation the successive coefficients $a_4, a_5, \dots$ by setting $x = x_0$.

Example. Determine $\phi''(0)$, $\phi'''(0)$, and $\phi^{iv}(0)$ if $y = \phi(x)$ is a solution of the initial value problem

$$y'' + xy' + \ln(x+1)y = 0, \quad y(0) = 1, \quad y'(0) = 2.$$

Solution.

Note that the important property that we used in determining the coefficients a_n was that we could compute infinitely many derivatives of $p(x)$ and $q(x)$. It might seem reasonable to relax the assumption that p and q are rational functions and simply require that they be infinitely differentiable in the neighborhood of x_0 . Unfortunately, this condition is too weak to ensure that we can prove the convergence of the resulting series expansion for $y = \phi(x)$. What is needed is to assume that the functions p and q are analytic at x_0 .

Definition. (Analytic Functions)

A function $f(x)$ is said to be analytic at a point $x = x_0$ if $f(x)$ has a Taylor series expansion

$$f(x) = \sum_{n=0}^{\infty} f_n(x - x_0)^n$$

that converge to it in some interval about x_0 .

Definition. (Ordinary Points)

If $p(x)$ and $q(x)$ are analytic functions at point x_0 , then x_0 is called an ordinary point of the equation

$$y'' + p(x)y' + q(x)y = 0.$$

If x_0 is not an ordinary point, then it is called a singular point.

Example. Show that $x = 0$ is an ordinary point of the equation

$$y'' + e^x y' + 3(x + 2)y = 0.$$

Solution.

Theorem 1. Let x_0 be an ordinary point of the equation

$$y'' + p(x)y' + q(x)y = 0.$$

Then

(1) the general solution of the equation is given by

$$y(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n = c_1 y_1(x) + c_2 y_2(x),$$

where a_0 and a_1 are arbitrary constants and y_1 and y_2 are linearly independent series solutions that are analytic at x_0 .

(2) the radius of convergence of each series solution y_1 and y_2 is at least as large as the minimum of the radii of convergence of the series of $p(x)$ and $q(x)$.

Lower bound for the radius of convergence

Let ρ_1 be the radius of convergence of $p(x) = \sum_{n=0}^{\infty} p_n(x - x_0)^n$ and let ρ_2 be the radius of convergence of $q(x) = \sum_{n=0}^{\infty} q_n(x - x_0)^n$. Then the radius of convergence of the series solution $y(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n$ of

$$y'' + p(x)y' + q(x)y = 0$$

is $\rho \geq \min\{\rho_1, \rho_2\}$. We can use the following proposition to find ρ_1 and ρ_2 .

Proposition 1. Let $f(x) = \frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials with no common factors.

Then the radius of convergence of the power series $f(x) = \sum_{n=0}^{\infty} f_n(x - x_0)^n$ is the distance from x_0 to the nearest zero of $Q(z)$.

Remark. In applying the above proposition we must consider real and complex roots of $Q(x)$.

Example 1. Determine a lower bound for the radius of convergence of the series solution of

$$(9 - x^2)y'' + 2y = 0$$

about $x = 1$.

Solution.

Example 2. Find a lower bound for the radius of convergence of the series solution of

$$(x - 2)(x^2 + 1)y'' - 4xy' + 6y = 0$$

about $x = 0$.

Solution.

Example 3. Find a lower bound for the radius of convergence of the series solution of

$$(x^2 + 2x + 5)y'' + 6xy' + 4x^2y = 0$$

about $x = 0$.

Solution.

5.4 Regular Singular Points

In this section we consider the equation

$$P(x)y'' + Q(x)y' + R(x)y = 0, \tag{9}$$

in the neighborhood of a singular point x_0 . Recall that if $P(x)$, $Q(x)$, and $R(x)$ are polynomials having no common factors, the singular points of the above equation are the points for which $P(x) = 0$.

Remarks.

- (1) If x_0 is a singular point of equation (9), then we can not use the method of Section 5.2 to find series solution about $x = x_0$.
- (2) Without additional information about the behavior of $\frac{Q(x)}{P(x)}$ and $\frac{R(x)}{P(x)}$ near x_0 , it is impossible to describe the behavior of the solutions of equation (9) near x_0 .

Definition. (Regular Singular Points)

Let $P(x)$, $Q(x)$, and $R(x)$ be polynomials with no common factors. A singular point x_0 of the equation

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

is called a regular singular point of the equation if

$$\lim_{x \rightarrow x_0} (x - x_0) \frac{Q(x)}{P(x)} \quad \text{and} \quad \lim_{x \rightarrow x_0} (x - x_0)^2 \frac{R(x)}{P(x)}$$

are finite. If a singular point is not regular, it is called irregular singular point.

Example 1. Find the singular points of the equation

$$x(x+2)y'' + (x-1)y' + 5y = 0$$

and determine whether they are regular or irregular.

Solution.

Example 2. Find all singular points of the equation

$$x^2(x-1)y'' + (x-2)y' - 3xy = 0$$

and classify them.

Solution.

Example 3. Show that the equation $xy'' + y' + y = 0$ has only one nonzero solution of the form $y = \sum_{n=0}^{\infty} a_n x^n$.

Solution.

5.5 Euler Equations

The simplest equation with a regular singular point at $x = 0$ is the Cauchy-Euler equation, or the equidimensional equation

$$L[y] = x^2 y'' + \alpha x y' + \beta y = 0,$$

where α and β are constants.

In this equation $P(x) = x^2$, $Q(x) = \alpha x$, and $R(x) = \beta$. Thus,

$$\lim_{x \rightarrow 0} x \frac{Q(x)}{P(x)} = \alpha \quad \text{and} \quad \lim_{x \rightarrow 0} x^2 \frac{R(x)}{P(x)} = \beta$$

and hence $x = 0$ is a regular singular point.

The solution of the Cauchy-Euler equation is typical of solutions of all differential equations with a regular singular point. Therefore, it is useful to study the solutions of this equation in detail.

In any interval not including the origin, the equation $L[y] = 0$ has a general solution of the form

$$y = c_1 y_1(x) + c_2 y_2(x),$$

where y_1 and y_2 are linearly independent solutions.

Let $y(x) = y(t)$, $t = \ln |x|$. Then

$$\begin{aligned} \frac{dt}{dx} &= \frac{1}{x}. \\ y' &= \frac{dy}{dt} \frac{dt}{dx} = \frac{1}{x} \frac{dy}{dt}, \\ y'' &= -\frac{1}{x^2} \frac{dy}{dt} + \frac{1}{x^2} \frac{d^2 y}{dt^2}. \end{aligned}$$

Substituting into the equation

$$L[y] = x^2 y'' + \alpha x y' + \beta y = 0$$

yields a second order linear equation with constant coefficients

$$\frac{d^2 y}{dt^2} + (\alpha - 1) \frac{dy}{dt} + \beta y = 0.$$

The characteristic equation reads $r^2 + (\alpha - 1)r + \beta = 0$.

Real Distinct Roots

If the roots r_1 and r_2 are real and distinct, then the solution is

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$

Thus

$$y(x) = c_1 |x|^{r_1} + c_2 |x|^{r_2}.$$

Proposition 2. *If the characteristic equation $r^2 + (\alpha - 1)r + \beta = 0$ has two real distinct roots r_1 and r_2 , then the general solution of the differential equation*

$$x^2 y'' + \alpha x y' + \beta y = 0$$

is given by

$$y(x) = c_1 |x|^{r_1} + c_2 |x|^{r_2},$$

where c_1 and c_2 are arbitrary constants.

Example 1. Solve $x^2y'' - 4xy' + 4y = 0$.

Solution.

Example 2. Solve $x^2y'' + 7xy' + 5y = x$, $x > 0$.

Solution.

Equal Roots

If the roots of $r^2 + (\alpha - 1)r + \beta = 0$ are equal, we have

$$y(t) = c_1 e^{r_1 t} + c_2 t e^{r_1 t}.$$

It follows that

$$y(x) = c_1 |x|^{r_1} + c_2 |x|^{r_1} \ln |x|.$$

Proposition 3. If the characteristic equation $r^2 + (\alpha - 1)r + \beta = 0$ has two equal real $r_1 = r_2$, then the general solution of the differential equation

$$x^2y'' + \alpha xy' + \beta y = 0$$

is given by

$$y(t) = c_1 |x|^{r_1} + c_2 |x|^{r_1} \ln |x|,$$

where c_1 and c_2 are arbitrary constants.

Example 1. Solve $x^2y'' - 3xy' + 4y = 0$.

Solution.

Example 2. Solve the initial value problem $x^2 y'' + 3xy' + y = 0$, $y(-1) = 1$, $y'(-1) = 2$.

Solution.

Complex Roots

If the roots of $r^2 + (\alpha - 1)r + \beta = 0$ are complex numbers

$$r_1 = \lambda + i\mu, \quad r_2 = \lambda - i\mu,$$

then we have

$$y(t) = c_1 e^{\lambda t} \cos(\mu t) + c_2 e^{\lambda t} \sin(\mu t).$$

It follows that

$$y(x) = c_1 |x|^\lambda \cos(\mu \ln |x|) + c_2 |x|^\lambda \sin(\mu \ln |x|).$$

Proposition 4. If the characteristic equation $r^2 + (\alpha - 1)r + \beta = 0$ has complex roots $r_1 = \lambda + i\mu$ and $r_2 = \lambda - i\mu$, then the general solution of the differential equation $x^2 y'' + \alpha x y' + \beta y = 0$ is given by

$$y(x) = c_1 |x|^\lambda \cos(\mu \ln |x|) + c_2 |x|^\lambda \sin(\mu \ln |x|),$$

where c_1 and c_2 are arbitrary constants.

Example 1. Solve $x^2y'' + 3xy' + 5y = 0$.

Solution.

Example 2. Solve $2x^2y'' + 2xy' + 8y = 1$.

Solution.

Example 3. Solve $x^2y'' + xy' + 4y = \sin(\ln x)$, $x > 0$.

Solution.

General Form of Euler Equation

The solution of an Euler equation of the form

$$(x - x_0)^2 y'' + \alpha(x - x_0)y' + \beta y = 0$$

can be obtained in similar way. We take $t = \ln |x - x_0|$ and the characteristic equation is the same $r^2 + (\alpha - 1)r + \beta = 0$.

Example 1. Solve $(x + 1)^2 y'' + 3(x + 1)y' + \frac{3}{4}y = 0$.

Solution.

Example 2. Solve $(3x - 1)^2 y'' + (9x - 3)y' - 9y = 0$.

Solution.