

Chapter 6

The Laplace Transform

Many practical engineering problems involve mechanical or electrical systems acted on by discontinuous or impulsive forcing terms. For such problems the methods described in Chapter 3 are often rather awkward to use. Another method that is especially well suited to these problems, although useful much more generally, is based on the Laplace transform.

The Laplace transform is named after mathematician and astronomer Pierre-Simon Laplace, who used a similar transform (now called the z -transform) in his work on probability theory. However, the techniques described in this chapter were not developed until a century or more later. They are due mainly to Oliver Heaviside (1850–1925), an innovative but unconventional English electrical engineer, who made significant contributions to the development and application of electromagnetic theory.

The Laplace transforms are simple so they are frequently used to solve a wide class of linear differential equations. Like other transforms, Laplace transforms are used to determine particular solutions.

6.1 Definition of the Laplace Transform

An integral transform is a relation of the form

$$F(s) = \int_{\alpha}^{\beta} \mathcal{K}(s, t) f(t) dt,$$

where a given function f is transformed into another function F by means of an integral. The function F is called the transform of f , and the function $\mathcal{K}$ is called the kernel of the transformation. The general idea in using an integral transform to solve a differential equation is as follows: Use the transformation to transform a problem for an unknown function f into a simpler problem for F , then solve this simpler problem to find F , and finally recover the desired function f from its transform F . This last step is known as “inverting the transform.”

By making a suitable choice of the kernel $\mathcal{K}$ and the integration limits α and β , it is possible to simplify substantially a problem involving a differential equation. Several integral transformations are widely used, each being appropriate for certain types of problems.

Definition. (Laplace Transform)

Let $f(t)$ be a function that is defined for $t \geq 0$. The Laplace transform of f is

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} f(t)e^{-st} dt = F(s)$$

for all values of the complex number s for which the improper integral converges.

Example 1. Find the Laplace transform of $f(t) = 1$.

Solution.

Example 2. Find the Laplace transform of $f(t) = t$.

Solution.

Example 3. Find the Laplace transform of $f(t) = e^{at}$, where a is a constant.

Solution.

Example 4. Find the Laplace transform of $f(t) = \sin(at)$, where a is a constant.

Solution.

Theorem 1. Let f be piecewise continuous in the interval $[0, A]$ for every $A > 0$. Assume that there exist positive real constants $K > 0$, $M > 0$ and a such that $|f(t)| \leq Ke^{at}$ for $t \geq M$. Then, the Laplace transform of $f(t)$ exists for $s > a$.

Theorem 2. (The Laplace transform is linear)

Let f and g be functions with Laplace transforms $F(s)$ and $G(s)$ respectively. Then for any real numbers a and b we have $\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\} = aF(s) + bG(s)$.

Proof. By linearity of integration. □

6.2 Solution of Initial Value Problems

In this section we show how the Laplace transform can be used to solve initial value problems for linear differential equations with constant coefficients. The usefulness of the Laplace transform in this connection rests primarily on the fact that the transform of f' is related in a simple way to the transform of f . The relationship is expressed in the following theorem.

Theorem 3. (Differentiation)

Suppose that f is continuous and f' is piecewise continuous on any interval $[0, A]$. Suppose that there are positive real constants K , M and a such that $|f(t)| \leq Ke^{at}$, for $t \geq M$. Then $\mathcal{L}\{f'(t)\}$ exists for $s > a$ and

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0).$$

Proof. Integration by parts. □

Corollary 1. Let $f, f', \dots, f^{(n-1)}$ be continuous and let $f^{(n)}$ be piecewise continuous on any interval $[0, A]$. Assume that there are positive real constants K , M and a such that $|f^{(k)}(t)| \leq Me^{at}$, $k = 0, 1, \dots, n-1$, for $t \geq M$. Then $\mathcal{L}\{f^{(n)}(t)\}$ exists for $s > a$ and

$$\mathcal{L}\{f^{(n)}(t)\} = s^n \mathcal{L}\{f(t)\} - s^{n-1}f(0) - \dots - sf^{(n-2)}(0) - f^{(n-1)}(0).$$

Example 1. Use $\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$ to find the $\mathcal{L}\{e^{2t}\}$.

Solution.

Example 2. Find the Laplace transform $Y(s) = \mathcal{L}\{y(t)\}$ of the solution of the initial value problem

$$y'' + 3y' = e^t, \quad y(0) = 1, \quad y'(0) = 2.$$

Solution.

Inverse Laplace Transform

Definition. (Inverse Laplace Transform)

Let $F(s)$ be the Laplace transform of a function $f(t)$ that is defined for $t > 0$. The inverse Laplace transform of $F(s)$ is

$$\mathcal{L}^{-1}\{F(s)\} = f(t).$$

The calculation of $\mathcal{L}^{-1}\{F(s)\}$ requires a knowledge of functions of complex variables. However, for some problems of mathematical physics, we need not use these calculation. We can avoid its use by expanding a given transform by the method of partial fractions in terms of simple fractions in the transform variables. With these simple functions, we refer to the table of Laplace transforms to obtain the inverse transforms. Here, we should note that we use the assumption that there is essentially a one-to-one correspondence between functions and their Laplace transforms. This may be stated as follows:

Theorem 4. (One-to-one correspondence between functions and their Laplace transforms)

Let f and g be piecewise continuous functions of exponential order. If there exists a constant s_0 such that $\mathcal{L}\{f\} = \mathcal{L}\{g\}$ for all $s > s_0$, then $f(t) = g(t)$ for all $t > 0$ except possibly at the points of discontinuity.

Theorem 5. (The inverse Laplace transform is linear)

Let f and g be functions with Laplace transforms $F(s)$ and $G(s)$ respectively. Then for any real numbers a and b we have $\mathcal{L}^{-1}\{aF(s) + bG(s)\} = a\mathcal{L}^{-1}\{F(s)\} + b\mathcal{L}^{-1}\{G(s)\}$.

Proof. By linearity of integration. □

Example 1. Find the inverse Laplace transform of $F(s) = \frac{5}{s-3}$.

Solution.

Example 2. Find the inverse Laplace transform of $F(s) = \frac{2s-3}{s^2+2s+10}$.

Solution.

Solving initial value problems

Assume that an initial problem is given for a function $y(t)$ with $t \in [0, \infty)$. Then there are three basic steps in solving the problem by the Laplace transform:

- (1) Apply the Laplace transform to the equation.
- (2) Solve the transformed problem to find the Laplace transform $Y(s) = \mathcal{L}\{y\}$.
- (3) Find the inverse Laplace transform of $Y(s)$.

Example 1. Use the Laplace transform to solve the initial value problem

$$y'' + y = \cos(3t), \quad y(0) = 1, \quad y'(0) = 2.$$

Solution.

Example 2. Use the Laplace transform to solve

$$y^{(4)} - 4y''' + 6y'' - 4y' + y = 0, \quad y(0) = 0, \quad y'(0) = 1, \quad y''(0) = 0, \quad y'''(0) = 1.$$

Solution.

Example 3. Use the Laplace transform to solve the initial value problem

$$y'' - 2y' + 2y = 0, \quad y(0) = 2, \quad y'(0) = 1.$$

Solution.

Example 4. Use the Laplace transform method to solve

$$y''' + y'' + 4y' + 4y = -2, \quad y(0) = 1, \quad y'(0) = 1, \quad y''(0) = -1.$$

Solution.

Example 5. Find the Laplace transform $Y(s) = \mathcal{L}\{y(t)\}$ of the solution of the initial value problem

$$y'' + 4y = \begin{cases} 1, & 0 \leq t < \pi \\ 0, & x \geq \pi. \end{cases}, \quad y(0) = 1, \quad y'(0) = 2.$$

Solution.

6.3 Step Functions

Some of the most interesting applications of the transform methods occur in the solution of linear differential equations with discontinuous or impulsive forcing functions. Equations of this type frequently arise in the analysis of the flow of current in electric circuits or the vibrations of mechanical systems. In this section and the following ones we develop some additional properties of the Laplace transform that are useful in the solution of such problems.

To deal effectively with functions having jump discontinuities, it is very useful to introduce the Heaviside or unit step function.

Definition. (Unit Step Function)

Let $c \geq 0$ be a constant. The function

$$u_c(t) = \begin{cases} 0, & t < c \\ 1, & t \geq c, \end{cases}$$

is called the Heaviside function or unit step function.

Example. Sketch the graph of $f(t) = u_1(t) - u_2(t)$.

Solution.

Functions related to u_c

For a given function f , defined for $t \geq 0$, we will often want to consider the related function g defined by

$$g(t) = u_c(t)f(t-c) = \begin{cases} 0, & t < c \\ f(t-c), & t \geq c. \end{cases}$$

The Laplace Transform of u_c

Theorem 6.

(a) $\mathcal{L}\{u_c(t)\} = \frac{1}{s}e^{-sc}, \quad s > 0.$

(b) If $c > 0$ is a constant, then $\mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}\mathcal{L}\{f(t)\}.$

(c) If $f(t) = \mathcal{L}^{-1}\{F(s)\}$, then $\mathcal{L}^{-1}\{e^{-cs}F(s)\} = u_c(t)f(t-c).$

Proof.

□

Example 1. Find $\mathcal{L}\{f(t)\}$ if $f(t) = \begin{cases} 0, & t < 1 \\ t-1, & t \geq 1. \end{cases}$

Solution.

Example 2. Find $\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s^2 + s - 2}\right\}.$

Solution.

Example 3. Find $\mathcal{L}\{f(t)\}$ if $f(t) = \begin{cases} t^2, & t < 2 \\ 0, & t \geq 2. \end{cases}$

Solution.

Example 4. Find $\mathcal{L}^{-1} \left\{ \frac{1}{s} [e^{-s} + e^{-3s} - e^{-6s}] \right\}$.

Solution.

Theorem 7. *Let c be a constant.*

(a) *If $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > a \geq 0$, then*

$$\mathcal{L}\{e^{ct}f(t)\} = F(s - c), \quad s > a + c.$$

(b) *If $f(t) = \mathcal{L}^{-1}\{F(s)\}$, then*

$$\mathcal{L}^{-1}\{F(s - c)\} = e^{ct}f(t).$$

Example 1. Find $\mathcal{L}^{-1}\left\{\frac{1}{s^2 - 6s + 13}\right\}$.

Solution.

Example 2. Find $\mathcal{L}\{e^{4t} \cos t\}$.

Solution.

6.4 Differential Equations with Discontinuous Forcing Functions

In this section we consider some examples in which the nonhomogeneous term, or forcing function, is discontinuous.

Example 1. Find the solution of the initial value problem

$$y'' + 2y' + 5y = g(t), \quad y(0) = 0, \quad y'(0) = 0,$$

$$\text{where } g(t) = \begin{cases} 1, & t < \pi, \\ 0, & t \geq \pi. \end{cases}$$

Solution.

Example 2. Solve

$$y'' + y = g(t), \quad y(0) = 2, \quad y'(0) = 3,$$

$$\text{where } g(t) = \begin{cases} t, & 0 \leq t < 1, \\ 2, & t \geq 1. \end{cases}$$

Solution.