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Islamic University of Gaza
Department of Math.
Summer semester 27/8/2013

Differential Equations
Final exam
Time: Two hours

	الاسم	Q1	Q2	Q3	Q4	Q5	Q6	Total
	الرقم	16	18	16	16	16	18	100
	المدرس							

Solve the following questions:

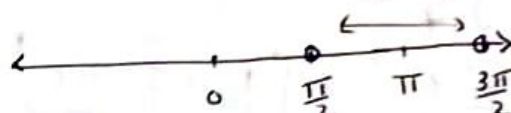
Q1) (a) [6 Marks] Determine the interval in which the solution of the given initial value problem exists

$$y' + (\tan t)y = \sin t, \quad y(\pi) = 0.$$

$$p(t) = \tan t = \frac{\sin t}{\cos t}$$

$$\mathbb{R} \setminus \left\{ \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \dots \right\}$$

$$g(t) = \sin t \quad \mathbb{R}$$



$$I = \left(\frac{\pi}{2}, \frac{3\pi}{2} \right)$$

b) [12 Marks] Suppose that $y_1(x) = x^2$ is one solution of the differential equation

$$x^2 y'' - 2y = 0.$$

Find a second linearly independent solution of the differential equation.

$$y = x^2 v$$

$$y' = 2xv + x^2 v'$$

$$y'' = 2v + 2xv' + 2xv' + x^2 v'' = 2v + 4xv' + x^2 v''$$

$$2x^2 v + 4x^3 v' + x^4 v'' - 2x^2 v = 0$$

$$x^4 v'' + 4x^3 v' = 0$$

$$x^4 z' + 4x^3 z = 0$$

v - missing

$$z = v'$$

$$z' = v''$$

$$\int \frac{dz}{z} = \int \frac{-4}{x} dx \Rightarrow \ln z = -4 \ln x + C_1$$

$$z = v' = C_1 x^{-4} \Rightarrow v = C_1 \frac{x^{-3}}{-3} + C_2$$

$$y = vx^2 = \frac{C_1}{-3x} + \frac{x^2 C_2}{1}$$

y_2

$$y_2 = \frac{1}{-3x}$$

$$y_c = C_1 e^{-2t} + C_2 t e^{-2t}$$

$$y_p = \left(\ln t + \frac{1}{t} e^{-4t} \right) e^{-2t} + \left(-\frac{1}{t} + \frac{1}{(n-1)t} e^{-4t} \right) t e^{-2t}$$

$$y = C_1 e^{-2t} + C_2 t e^{-2t} - e^{-2t} \ln t + \frac{1}{t} e^{-2t} - e^{-2t} + \frac{1}{(n-1)} e^{-2t}$$

Q2) (a) [10 Marks] Find the general solution of the differential equation

$$y'' + 4y' + 4y = t^{-2} e^{-2t} \quad [t > 0]$$

$$y = C_1 e^{-2t} + C_2 t e^{-2t} - \frac{1}{t} e^{-2t}$$

$$\text{ch. eq } r^2 + 4r + 4 = 0$$

$$(r+2)^2 = 0$$

repeat

$$r = -2$$

$$= \int \frac{y_1 g(t)}{w}$$

$$y_c = C_1 e^{-2t} + C_2 t e^{-2t}$$

$$y_p = u_1 e^{-2t} + u_2 t e^{-2t}$$

$$u_1 = \int \frac{w_1 g(t)}{w} = \int \frac{e^{-2t} \cdot e^{-2t}}{t^2} = \int \frac{e^{-4t}}{t^2} dt$$

$$= \int \frac{y_2 g(t)}{w}$$

$$u_2 = \int \frac{w_2 g(t)}{w} = \int \frac{e^{-4t}}{t^2} dt$$

$$\Rightarrow \int \frac{e^{-4t}}{t^2} dt = \int \sum_{n=0}^{\infty} \frac{(-4)^n t^{n-2}}{n!} = \int \frac{1}{t^2} + \sum_{n=1}^{\infty} \frac{(-4)^n t^{n-2}}{n!}$$

$$= -\frac{1}{t} + \sum_{n=1}^{\infty} \frac{(-4)^n t^{n-1}}{(n-1)n!} + K = \left(-\frac{1}{t} \right) + \frac{1}{(n-1)t} e^{-4t} + K_2$$

(b) [6 Marks] Solve the following differential equation

$$\frac{dy}{dx} = \frac{3y^2 - x^2}{2xy} \quad \therefore \frac{y^2}{x}$$

$$\text{homog. } v = \frac{y}{x}$$

$$y = vx$$

$$y' = v + xv'$$

$$v + xv' = \frac{3v^2 - 1}{2v}$$

$$xv' = \frac{3v^2 - 1 - 2v}{2v}$$

$$v = \frac{2 \pm \sqrt{4+4x^3}}{6}$$

$$= \frac{1}{3} \pm \frac{2}{3} \sqrt{x^3}$$

$$\frac{2v}{3v^2 - 2v - 1} dv = \frac{1}{x} dx$$

$$\int \frac{2v}{(v+\frac{1}{3})(v-1)} dv = \int \frac{1}{x} dx$$

$$\frac{1}{2} \ln |v+\frac{1}{3}| + \frac{3}{2} \ln |v-1| = \ln |x| + C$$

$$\frac{1}{2} \ln |\frac{y}{x} + \frac{1}{3}| + \frac{3}{2} \ln |\frac{y}{x} - 1| = \ln |x| + C$$

$$\frac{2v}{(v+\frac{1}{3})(v-1)} = \frac{A}{(v+\frac{1}{3})} + \frac{B}{v-1}$$

$$2v = A(v-1) + B(v+\frac{1}{3})$$

$$\Rightarrow \boxed{v = -\frac{1}{3}}$$

$$\boxed{v=1}$$

$$\frac{2}{3} = -\frac{4}{3}A \Rightarrow \boxed{A = \frac{1}{2}}$$

$$\frac{2}{3} = \frac{4}{3}B \Rightarrow \boxed{B = \frac{3}{2}}$$

$y_1 = c_1 e^{-2t} + c_2 t e^{-2t}$
 $y_p = (ln t + \frac{1}{t} e^{-2t}) e^{-2t} + (-\frac{1}{t} + \frac{1}{(t-1)} \frac{t}{t}) t e^{-2t}$
 $y = c_1 e^{-2t} + c_2 t e^{-2t} - e^{-2t} \ln t + \frac{1}{t} e^{-2t} - e^{-2t} + \frac{1}{(t-1)} e^{-2t}$

Q2) (a) [10 Marks] Find the general solution of the differential equation

$y'' + 4y' + 4y = t^{-2} e^{-2t} \quad [t > 0]$

$char. \quad r^2 + 4r + 4 = 0$

المشكلة

$(Q_2) a) \quad y'' + 4y' + 4y = t^{-2} e^{-2t} \quad [t > 0]$

$y_c = c_1 e^{-2t} + c_2 t e^{-2t}$
 $y_p = u_1 e^{-2t} + u_2 t e^{-2t}$

$w(y_1, y_2) = 1$

$* u_1 = \int \frac{-y_2 g(t)}{w(y_1, y_2)} dt = \int \frac{-t e^{-2t} \cdot e^{-2t}}{t^2} dt$
 $= \int \frac{-e^{-4t}}{t} dt = \int \frac{-1}{t} dt = -\ln t + k_1$

$\Rightarrow e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow \frac{e^{-4t}}{t} = \sum_{n=0}^{\infty} \frac{(-4)^n t^{n-1}}{n!}$
 $= t^{-1} + \sum_{n=1}^{\infty} \frac{(-4)^n t^{n-1}}{n!}$

$* u_2 = \int \frac{y_1 g(t)}{w(y_1, y_2)} dt = \int \frac{e^{-4t}}{t^2} dt = \int \frac{1}{t^2} dt = -\frac{1}{t} + k_2$

$\Rightarrow \frac{e^{-4t}}{t^2} = \sum_{n=0}^{\infty} \frac{(-4)^n t^{n-2}}{n!} = \left(\frac{1}{t^2} \right) + \sum_{n=1}^{\infty} \frac{(-4)^n t^{n-2}}{n!}$

$y_p = (-\frac{1}{t} \ln t) e^{-2t} + (-\frac{1}{t}) t e^{-2t} + k$

$y = c_1 e^{-2t} + c_2 t e^{-2t} - \ln t e^{-2t}$

$v = \frac{2 \pm \sqrt{4}}{6}$
 $= \frac{1}{3} \text{ or } 1$

Q3) a) [10 Marks] Consider the differential equation

$$y^{(4)} + y''' = 3t + 4e^{-t}.$$

(i) [5 Marks] Find the solution of the corresponding homogeneous equation.

Ch. eq $r^4 + r^3 = 0 \Rightarrow r^3(r+1) = 0$
 repeat $\boxed{r=0}$ $\boxed{r=-1}$

$$y_c = C_1 + C_2 t + C_3 t^2 + C_4 e^{-t}$$

(ii) [5 Marks] Use the method of undetermined coefficients to find a suitable form of the particular solution without evaluating the constants.

$$y_{p1} = (At + B)t^3 = At^4 + Bt^3$$

$$y_{p2} = Dte^{-t}$$

$$y_p = At^4 + Bt^3 + Dte^{-t}$$

(b) [6 Marks] Solve the differential equation

$$x^2 y'' - 4xy' + 4y = 0.$$

Ch. eq $r(r-1) - 4r + 4 = 0$

$$r^2 - 5r + 4 = 0$$

$$(r-4)(r-1) = 0$$

$$r=4 \quad r=1$$

$$y = C_1 x^4 + C_2 x \quad x > 1$$

$$y = C_1 |x|^4 + C_2 |x| \quad x < 1$$

Q4) [16 Marks] Find two linearly independent power series solutions at $x_0 = 0$ to the differential equation (include at least three nonzero terms in each series)

$$y'' + xy' + 2y = 0.$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^n + 2 \sum_{n=0}^{\infty} a_n x^n = 0$$

$n \rightarrow n+2$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=1}^{\infty} n a_n x^n + 2 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$2a_2 + 2a_0 + \sum_{n=1}^{\infty} ((n+2)(n+1)a_{n+2} + n a_n + 2a_n) x^n = 0$$

$$\boxed{a_2 = -a_0}$$

$$\boxed{n=1} \quad 6a_3 + a_1 + 2a_1 = 0 \quad \rightarrow a_3 = -\frac{a_1}{2}$$

$$\boxed{n=2} \quad 12a_4 + 2a_2 + 2a_2 = 0 \quad \rightarrow a_4 = -\frac{a_2}{3} = \frac{a_0}{3}$$

$$\boxed{n=3} \quad 20a_5 + 3a_3 + 2a_3 = 0 \quad \rightarrow a_5 = -\frac{a_3}{4} = \frac{a_1}{8}$$

$$y = \sum_{2n=0}^{\infty} \frac{(-1)^{2n} a_0 x^{2n}}{(2n+1)} + \sum_{2n-1=0}^{\infty} \frac{(-1)^{2n-1} a_1 x^{2n-1}}{2n}$$

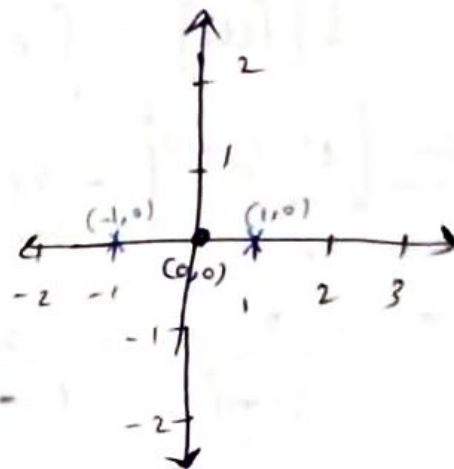
$$y = a_0 \left(+1 - x^2 + \frac{x^4}{3} - \dots \right) + a_1 \left(x - \frac{x^3}{2} + \frac{x^5}{8} - \dots \right)$$

Q5) (a) [6 Marks] Determine a lower bound for the radius of convergence of the series solution for the given differential equation about $x_0 = 0$

$$(1-x^2)y'' + 3xy' + y = 0.$$

$$P(x) = \frac{3x}{1-x^2} \quad \mathbb{R} \setminus \{-1, 1\}$$

$$Q(x) = \frac{1}{1-x^2} \quad \mathbb{R} \setminus \{-1, 1\}$$



$$L = \sqrt{(1-0)^2 + (0-0)^2} = 1$$

$$\boxed{R_{\min} = 1}$$

$$L = \sqrt{(-1-0)^2 + (0-0)^2} = 1$$

(b) [10 Marks] Determine the singular points of the differential equation

$$2x(x-2)^2 y'' + 3xy' + (x-2)y = 0,$$

and classify them as regular or irregular.

$$2x(x-2)^2 = 0$$

$$x=0$$

$$x=2$$

$$\boxed{x=0}$$

$$\lim_{x \rightarrow 0} x \cdot \frac{3x}{2x(x-2)^2} = 0$$

$$\lim_{x \rightarrow 0} x^2 \cdot \frac{(x-2)}{2x(x-2)^2} = 0$$

regular

$$\boxed{x=2}$$

$$\lim_{x \rightarrow 2} (x-2) \cdot \frac{3x}{2x(x-2)^2} \begin{matrix} \nearrow \infty \\ \searrow -\infty \end{matrix}$$

d.n.e.

irregular

Q6) (a) [6 Marks] Use the definition to find the Laplace transformation of the function

$$L\{f(t)\} = \int_0^{\infty} e^{-st} \cdot t^2 e^t dt \quad f(t) = t^2 e^t = \lim_{b \rightarrow \infty} \int_0^b t^2 \cdot e^{(-s+1)t} dt$$

$$= \lim_{b \rightarrow \infty} \left[\frac{t^2}{(-s+1)} e^{(-s+1)t} - \frac{2t}{(-s+1)^2} e^{(-s+1)t} + \frac{2}{(-s+1)^3} e^{(-s+1)t} \right]_0^b$$

$$= \frac{-2}{-(s+1)^3} = \frac{2}{(s+1)^3}$$

$n=2$ $t^2 e^t$ $\checkmark$ مشتق

Diagram showing the integration by parts steps for $\int t^2 e^{(-s+1)t} dt$:

- Let $u = t^2$, $dv = e^{(-s+1)t}$
- $du = 2t$, $v = \frac{e^{(-s+1)t}}{(-s+1)}$
- $\int t^2 e^{(-s+1)t} dt = \frac{t^2}{(-s+1)} e^{(-s+1)t} - \int \frac{2t}{(-s+1)} e^{(-s+1)t} dt$
- $= \frac{t^2}{(-s+1)} e^{(-s+1)t} - \frac{2}{(-s+1)^2} e^{(-s+1)t} + \frac{2}{(-s+1)^3} e^{(-s+1)t}$
- $e^{-\infty} = 0$

(b) [12 Marks] Use the Laplace transform to solve the initial value problem

$$y'' - 2y' + 2y = \cos t, \quad y(0) = 1, \quad y'(0) = 0.$$

$$s^2 Y(s) - s y(0) - y'(0) - 2s Y(s) + 2 y(0) + 2 Y(s) = L\{\cos t\} = \frac{s}{s^2 + 1}$$

$$Y(s) [s^2 - 2s + 2] - s + 2 = \frac{s}{s^2 + 1}$$

$$Y(s) = \frac{s-2}{(s-1)^2 + 1} + \frac{s}{(s^2 + 1)(s^2 + 2s + 2)}$$

$$Y(s) = \frac{s-1}{(s-1)^2 + 1} - \frac{1}{(s-1)^2 + 1} + \frac{1}{5} \frac{s}{s^2 + 1} + \frac{2}{5} \cdot \frac{1}{s^2 + 1} + \frac{-1}{5} \frac{(s-1)}{(s-1)^2 + 1} - \frac{1}{(s-1)^2 + 1}$$

$$= e^t \cos t - e^t \sin t + \frac{1}{5} \cos t + \frac{2}{5} \sin t - \frac{1}{5} e^t \cos t - e^t \sin t$$

$$= \frac{4}{5} e^t \cos t - 2e^t \sin t + \frac{1}{5} \cos t + \frac{2}{5} \sin t$$

$$\frac{s}{(s^2 + 1)(s^2 + 2s + 2)} = \frac{As + B}{s^2 + 1} + \frac{Cs + D}{s^2 + 2s + 2}$$

$$s = (As + B)(s^2 + 2s + 2) + (Cs + D)(s^2 + 1)$$

$$A + C = 0$$

$$2A + B + D = 0$$

$$\frac{2}{5} 2A + 2B + C = 1 \Rightarrow 2B = \frac{4}{5} \Rightarrow B = \frac{2}{5}$$

$$2B + D = 0 \Rightarrow 2B = -D \Rightarrow D = -\frac{4}{5}$$

$$B = \frac{2}{5}$$

$$D = -\frac{4}{5}$$

$$C = -\frac{1}{5}$$

$$A + C = 0$$

$$2A - B = 0$$

$$2A + 2B + C = 1$$

$$6A + C = 1$$

$$A + C = 0$$

$$5A = 1 \Rightarrow A = \frac{1}{5}$$

The Islamic University Of Gaza
Department of Mathematics
Ordinary Differential Equations (Math2302)

Date: 14/6/2004

Final Exam

Time: Two hours

Name: <u>Somaria</u>	Q.1	Q.2	Q.3	Q.4	Q.5	Q.6	Total
ID#:							
Instructor:	10	18	18	18	18	18	100

Answer the following questions:

Q.1 Choose the correct answer for each of the following:

(i) The ordinary differential equation $\left(\frac{dy}{dt}\right)^5 + \frac{d^4y}{dx^4} - t^3y^7 + x^7 = 0$ has order

- (a) 5 (b) 4 (c) 7

(ii) The differential equation $(t+1)^N dy + (3y+e^t)^M dt = 0$ is

- $y' + \frac{3}{t+1}y + \frac{e^t}{t+1} = 0$ (a) a linear equation (b) a Bernoulli equation (c) an exact equation.

$M_y = 3$ $N_t = 1$
not exact

(iii) The point $x = 1$ is

- (a) an ordinary (b) a regular singular (c) an irregular singular

point of the differential equation $(x^2 - 2x + 1)y'' + (x - 1)y' + y = 0$.

$(x-1)^2$

$$\lim_{x \rightarrow 1} (x-1) \cdot \frac{(x-1)}{(x-1)^2} = 1$$

$$\lim_{x \rightarrow 1} (x-1)^2 \cdot \frac{1}{(x-1)^2} = 1$$

regular

(iv) If y_1 is a solution of $y'' + p(t)y = 0$ and y_2 is a solution of $y'' + p(t)y = g(t)$, then $y = y_1 + y_2$ is a solution of

(a) $y'' + p(t)y = 0$

(b) $y'' + p(t)y = g(t)$

(c) both of $y'' + p(t)y = 0$ and $y'' + p(t)y = g(t)$

$y = y_c$

$y = y_p + y_c$

(v) To find a particular solution of $y''' + ty' + y = e^t$, we can use

(a) the method of variation of parameters.

(b) the method of undetermined coefficients.

(c) both of them.

Q.2 (a) Solve the differential equation

$$x^2 y' = y^2 + 4xy \quad (x^2)$$

homog.

$$v = \frac{y}{x}$$

$$y = vx$$

$$y' = v + x \frac{dv}{dx}$$

$$y' = \left(\frac{y}{x}\right)^2 + 4\left(\frac{y}{x}\right)$$

$$v + x \frac{dv}{dx} = v^2 + 4v$$

$$x \frac{dv}{dx} = v^2 + 3v$$

$$\int \frac{dv}{v(v+3)} = \int \frac{1}{x} dx$$

$$\int \left(\frac{1}{3} \frac{1}{v} + \frac{-1}{3} \cdot \frac{1}{v+3} \right) dv = \int \frac{1}{x} dx$$

$$\frac{1}{v(v+3)} = \frac{A}{v} + \frac{B}{v+3}$$

$$1 = A(v+3) + Bv$$

$$v=0 \quad 3A=1 \Rightarrow A=\frac{1}{3}$$

$$v=-3 \quad B=-\frac{1}{3}$$

$$\frac{1}{3} \ln |v| - \frac{1}{3} \ln |v+3| = \ln |x| + C$$

$$\frac{1}{3} \ln \left| \frac{y}{x} \right| - \frac{1}{3} \ln \left| \frac{y}{x} + 3 \right| = \ln |x| + C$$

(b) Find the values of k so that $y = x^k$ is a solution of the differential equation

$$x^3 y''' - 6xy' = 0$$

$$y = x^r$$

$$y' = r x^{r-1}$$

$$y'' = r(r-1)x^{r-2}$$

$$y''' = r(r-1)(r-2)x^{r-3}$$

$$r(r-1)(r-2) - 6r = 0$$

$$r(r^2 - 3r + 2) - 6r = 0$$

$$r^3 - 3r^2 + 2r - 6r = 0$$

$$r^3 - 3r^2 - 4r = 0$$

$$r(r^2 - 3r - 4) = 0$$

$$r=0, (r-4)(r+1)=0$$

$$r=4, r=-1$$

$$y = C_1 x^0 + C_2 x^4 + C_3 x^{-1}$$

$$\boxed{k = 0, 4, -1}$$

Q.3 Use the method of undetermined coefficients to determine a suitable form of a particular solution of the equation

$$y^{(5)} + 4y^{(4)} + y''' + 4y'' = 5 + te^{-2t} + \cos t$$

(DO NOT EVALUATE THE CONSTANTS)

ch. eq $r^5 + 4r^4 + r^3 + 4r^2 = 0$

$$r^2 \left[\underbrace{r^3 + 4r^2 + r + 4}_{\substack{\uparrow \\ r^2(r^2+1) + r + 4}} \right] = 0$$

$$r^2 \left[r(r^2+1) + 4(r^2+1) \right] = 0$$

$$r^2(r^2+1)(r+4) = 0$$

$\downarrow$ $r=0$ repeat
 $\downarrow$ $r=\pm i$
 $\downarrow$ $r=-4$

$$y_c = C_1 + C_2 t + C_3 \cos t + C_4 \sin t + C_5 e^{-4t}$$

$$y_p = \underbrace{At^2}_{y_{p1}} + \underbrace{(Bt+C)}_{y_{p2}} e^{-2t} + (K_1 \cos t + K_2 \sin t) * t$$

$$y_p = At^2 + Bte^{-2t} + Ce^{-2t} + K_1 t \cos t + K_2 t \sin t$$

Q.5 Find the general solution of the differential equation

$$x^2 y'' + xy' + 4y = 1, \quad (x > 0)$$

$$e^z = x$$

$$\ln x = z$$

$$\frac{d^2 y}{dz^2} + (1-1) \frac{dy}{dz} + 4y = 1$$

$$\text{Char eq. } r^2 + 4 = 0$$

$$r = \pm 2i$$

$$y_c = C_1 \cos 2z + C_2 \sin 2z$$

$$\ln y_c = C_1 \cos(2 \ln x) + C_2 \sin(2 \ln x)$$

$$y_p = A$$

$$y = y_c + y_p = C_1 \cos(2 \ln x) + C_2 \sin(2 \ln x)$$

(or)

$$r(r-1) + r + 4 = 0$$

$$r^2 + 4 = 0$$

$$r = \pm 2i$$

$$y_c = C_1 \cos 2 \ln x + C_2 \sin(2 \ln x)$$

$$y_p = A$$

$$A = \frac{1}{4}$$

$$y_p' = 0$$

$$y_p'' = 0$$

$$0 + 0 + 4A = 1$$

Q.6 (a) Show that the recurrence relation of the series solution of the differential equation $y'' - (x-2)y = 0$ about $x=2$ is given by $a_{n+3} = \frac{a_n}{(n+2)(n+3)}$, $n = 0, 1, 2, 3, \dots$

$$\sum_{n=2}^{\infty} \underbrace{n(n-1)a_n}_{(n=n+3)} (x-2)^{n-2} - \sum_{n=0}^{\infty} a_n (x-2)^{n+1} = 0$$

$$\sum_{n=1}^{\infty} (n+3)(n+2) a_{n+3} (x-2)^{n+1} - \sum_{n=0}^{\infty} a_n (x-2)^{n+1} = 0$$

$$2a_2 + \sum_{n=0}^{\infty} ((n+3)(n+2) a_{n+3} - a_n) (x-2)^{n+1} = 0$$

$$\boxed{a_2 = 0}$$

$$a_{n+3} = \frac{a_n}{(n+3)(n+2)} \quad \#$$

(b) Use part (a) to find the series solution of $y'' - (x-2)y = 0$ about $x=2$.

$$\boxed{n=0} \quad 6a_3 - a_0 = 0 \quad \rightarrow a_3 = \frac{a_0}{6} = \frac{a_0}{3!}$$

$$\boxed{n=1} \quad 12a_4 - a_1 = 0 \quad \rightarrow a_4 = \frac{a_1}{12} = \frac{a_1}{4!}$$

$$\boxed{n=2} \quad 20a_5 - a_2 = 0 \quad \rightarrow a_5 = 0$$

$$\boxed{n=3} \quad 30a_6 - a_3 = 0 \quad \rightarrow a_6 = \frac{a_3}{30} = \frac{a_0}{6 \times 5 \times 3 \times 2} = \frac{a_0}{6!}$$

$$\boxed{n=4} \quad 42a_7 - a_4 = 0 \quad \rightarrow a_7 = \frac{a_4}{42} = \frac{a_1}{7 \times 6 \times 3 \times 4}$$

$$y = a_0 \left(1 + \frac{x^3}{3!} + \frac{x^6}{6!} + \dots \right) + a_1 \left(x + \frac{x^4}{12} + \frac{x^7}{504} + \dots \right)$$



Islamic University Of Gaza
Department of Mathematics
Ordinary Differential Equations (Math 2302)

Final Exam
Date: 25/5/2017
Time: Two Hours

Somaiq	إسم الطالب :	Q.1	Q.2	Q.3	Q.4	Q.5	Q.6	Q.7	Total
	الرقم الجامعي :								
	المدرس :	14	14	14	14	15	15	14	100
Answer the following 7 questions:									

Q.1 (a) Solve the differential equation $y' = e^{2x-y}$.

$$\frac{dy}{dx} = y' = e^{2x} e^{-y}$$

$$\int e^y dy = \int e^{2x} dx$$

seperable

$$\ln e^y = \ln \frac{e^{2x}}{2} + C$$

$$y = \ln \left(\frac{e^{2x}}{2} + C \right)$$

(b) Find a function $f(x)$ with $f(2) = 0$ such that the differential equation

$$(xy + \sin x)dx + (xf(x) + y)dy = 0$$

is an exact equation.

$$M_y = x$$

$$N_x = f(x) + x f'(x)$$

$$x = f(x) + x f'(x)$$

$$\int (x f(x))' = \int x$$

$$x f(x) = \frac{x^2}{2} + C \Rightarrow f(x) = \frac{x}{2} + \frac{C}{x}$$

$$f(2) = 0 \Rightarrow f(2) = 1 + \frac{C}{2} = 0$$

$$\frac{C}{2} = -1 \Rightarrow \boxed{C = -2}$$

$$f(x) = \frac{x}{2} - \frac{2}{x}$$

Q.2 (a) Show that t , t^2 , and t^4 are fundamental solutions of $t^3 y''' - 4t^2 y'' + 8t y' - 8y = 0$, $t > 0$.

$y = t$ $y' = 1$ $y'' = 0$ $y''' = 0$ $0 - 0 + 8t - 8t = 0$ $\checkmark$	$y = t^2$ $y' = 2t$ $y'' = 2$ $y''' = 0$ $0 - 8t^2 + 16t^2 - 8t^2 = 0$ $\checkmark$	$y = t^4$ $y' = 4t^3$ $y'' = 12t^2$ $y''' = 24t$ $24t^4 - 48t^4 + 24t^4 - 8t^4$ $= -8t^4 \neq 0$
---	--	---

$$W(t, t^2, t^4) = \begin{vmatrix} t & t^2 & t^4 \\ 1 & 2t & 4t^3 \\ 0 & 2 & 12t^2 \end{vmatrix} = t[(24t^3) - 8t^3] - [12t^4 - 2t^4]$$

$$= 16t^4 - 10t^4 = 6t^4 \neq 0$$

(b) Find and classify the singular points of the differential equation

$$x^3(x+1)y'' + x(x-2)y' + 4y = 0.$$

$$x=0$$

$$x+1=0$$

$$x=-1$$

$$\boxed{x=0}$$

$$\lim_{x \rightarrow 0} x \cdot \frac{(x-2)}{x^2(x+1)} \begin{cases} \infty \\ -\infty \end{cases} \text{ d.n.e}$$

irregular

$$\boxed{x=-1}$$

$$\lim_{x \rightarrow -1} (x+1) \cdot \frac{(x-2)}{x^2(x+1)} = \boxed{-3}$$

$$\lim_{x \rightarrow -1} (x+1)^2 \cdot \frac{4}{x^3(x+1)} = 0$$

regular

Q.3 Find the general solution of the equation

$$y''' - 2y'' + y' - 2y = e^{2t} + \sin(2t).$$

$$y = e^{rt}$$

$$\text{Ch eq. } r^3 - 2r^2 + r - 2 = 0$$

$$r^2(r-2) + (r-2) = 0$$

$$(r-2)[r^2+1] = 0$$

$$\downarrow$$
$$r = 2$$

$$\downarrow$$
$$r = \pm i$$

$$y_c = C_1 e^{2t} + C_2 \cos t + C_3 \sin t$$

$$y_p = A t e^{2t} + B \sin 2t + C \cos 2t$$

$$y = y_c + y_p$$
$$= C_1 e^{2t} + C_2 \cos t + C_3 \sin t + A t e^{2t} + B \sin 2t + C \cos 2t$$

Q.4 Find the general solution of

$$x^2 y'' - 3xy' + 3y = x^4 \sin x, \quad \boxed{x > 0.}$$

$$y = x^r \quad \text{check.} \quad r(r-1) - 3r + 3 = 0$$

$$r^2 - 4r + 3 = 0$$

$$(r-3)(r-1) = 0 \quad r=3, \quad r=1$$

$$y_c = C_1 x^3 + C_2 x^1$$

$$y_p = (Ax^4 + Bx^3 + Cx^2 + Dx + E) [K_1 \sin x + K_2 \cos x]$$

(or)

$$z = x$$

$$z = \ln x$$

$$\frac{d^2 y}{dz^2} - 4 \frac{dy}{dz} + 3y = e^{4z} \sin(e^z)$$

$$\text{check } r^2 - 4r + 3 = 0$$

$$r=3, \quad r=1$$

$$y_c = C_1 e^{3z} + C_2 e^z \Rightarrow y_c = C_1 x^3 + C_2 x$$

$$y_p = e^{4z} [A \cos(e^z) + B \sin(e^z)]$$

$$= x^4 [A \cos(x) + B \sin x]$$

Q.5 Find the series solution (the first five nonzero terms) of the initial value problem

$$(x^2 + 1)y'' + xy' + 3xy = 0, \quad y(0) = 2, \quad y'(0) = 3.$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^n + \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^n + 3 \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

$\downarrow$ $\downarrow$
 $n = n+2$ $n = n-1$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^n + \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n + \sum_{n=1}^{\infty} n a_n x^n + 3 \sum_{n=1}^{\infty} a_{n-1} x^n = 0$$

$$2a_2 + 6a_3 x + a_1 x + 3a_0 x + \sum_{n=2}^{\infty} [n(n-1)a_n + (n+2)(n+1)a_{n+2} + n a_n + 3a_{n-1}] x^n = 0$$

$a_2 = 0$

$$6a_3 + a_1 + 3a_0 = 0 \Rightarrow a_3 = -\frac{a_1}{6} - \frac{a_0}{2} \Rightarrow a_3 = -\frac{1}{2} - 1 = -\frac{3}{2}$$

$$n=2 \quad (2a_2 + 2a_4 + 2a_2 + 3a_1)x^2 = 0 \quad a_4 = -\frac{3}{2}a_1$$

$$n=2 \quad 2a_2 + 12a_4 + 2a_2 + 3a_1 = 0 \quad a_4 = -\frac{a_1}{4} \Rightarrow a_4 = -\frac{3}{4}$$

$$n=3 \quad 6a_3 + 20a_5 + 3a_3 + 3a_2 = 0$$

$$n=4 \quad 12a_4 + 30a_6 + 4a_4 + 3a_3 = 0$$

$$a_5 = -\frac{9}{20}a_3 = \frac{3a_1}{40} + \frac{9a_0}{40}$$

$$a_5 = \frac{9}{40} + \frac{9}{20}$$

$$a_6 = -\frac{16a_4}{36} - \frac{3a_3}{36}$$

$$= \frac{2a_1}{15} + \frac{a_1}{160} + \frac{a_0}{20}$$

$$a_6 = \frac{2}{5} + \frac{3}{160} + \frac{1}{16}$$

$$a_6 = \frac{1}{2} + \frac{3}{160}$$

$$a_0 = 2$$

$$a_1 = 3$$

Q.6 Use the Laplace transform method to solve the initial value problem

$$y''' - 2y'' + y' = t, \quad y(0) = y'(0) = y''(0) = 2.$$

$$s^3 Y(s) - s^2 y(0) - s y'(0) - y''(0) - 2s^2 Y(s) + 2s y(0) + 2y'(0) + s Y(s) - y(0) = L\{t\} = \frac{1}{s^2}$$

$$Y(s) [s^3 - 2s^2 + s] - 2s^2 - 2s - 2 + 4s + 4 - 2 = \frac{1}{s^2}$$

$$Y(s) [s(s^2 - 2s + 1)] = 2s^2 - 2s + \frac{1}{s^2}$$

$$Y(s) = \frac{2s(s-1)}{s(s^2-2s+1)} + \frac{1}{s^3(s^2-2s+1)}$$

$$= \frac{2}{(s-1)} + \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{(s-1)} + \frac{K}{(s-1)^2}$$

$$= 2e^t + A + Bt + \frac{1}{2}t^2 + De^t + te^t$$

$$\frac{1}{s^3(s^2-2s+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{(s-1)} + \frac{K}{(s-1)^2}$$

$$1 = As^2(s-1)^2 + Bs(s-1)^2 + C(s-1)^2 + s^3(s-1)D + s^3K$$

$$s=0$$

$$C=1$$

$$s=-1$$

$$K=+1$$

$$s=-1$$

$$1 = 4A - 4B + 4C + 2D - K$$

$$1 = 4A - 4B + 2D + 3$$

$$-2 = 4A - 4B + 2D$$

$$-1 = 2A + 2B + D \quad (1)$$

$$s=2$$

$$1 = 4A + 2B + 1 + 8D + 8$$

$$-8 = 4A + 2B + 8D \quad (2)$$

$$-1 = 2A - 2B + D$$

$$-9 = 6A + 9D \quad (3)$$

Q.7 (a) Given that $y_1 = \sin t$ is a solution of the equation $y^{(4)} + 2y''' + 6y'' + 2y' + 5y = 0$, find the general solution of this equation.

$$r^4 + 2r^3 + 6r^2 + 2r + 5 = 0$$

$$(r^4 - 1) + (2r^3 + 2r) + (6r^2 + 6) = 0$$

$$(r^2 - 1)(r^2 + 1) + 2r(r^2 + 1) + 6(r^2 + 1) = 0$$

$$(r^2 + 1) [(r^2 - 1) + 2r + 6] = 0 \Rightarrow (r^2 + 1) [r^2 + 2r + 5] = 0$$

(3+1) ~~Find the roots~~

$$r = \pm i, \quad r = -1 \pm 2i \quad \begin{matrix} \lambda = -1 \\ \mu = 2 \end{matrix}$$

$$r = \frac{-2 \pm \sqrt{4 - 4 \times 5}}{2} = -1 \pm 2i$$

$\lambda = 0$
 $\mu = 1$

$$y = C_1 \cos t + C_2 \sin t + C_3 e^{-t} \cos 2t + C_4 e^{-t} \sin 2t$$

(b) Use the definition to find the Laplace transform of $f(t) = \begin{cases} t, & t < 1, \\ 1, & t \geq 1. \end{cases}$

$$L\{f(t)\} = \int_0^1 e^{-st} \cdot t \, dt + \int_1^\infty 1 \cdot e^{-st} \, dt$$

$$F(s) = \left(-\frac{t}{s} e^{-st} - \frac{e^{-st}}{s^2} \right) \Big|_0^1 + \lim_{b \rightarrow \infty} \int_1^b e^{-st} \, dt$$

$$= -\frac{e^{-s}}{s} - \frac{e^{-s}}{s^2} + \frac{1}{s^2} + \lim_{b \rightarrow \infty} \left(-\frac{e^{-sb}}{s} + \frac{e^{-s}}{s} \right)$$

$$= \cancel{\frac{e^{-s}}{s}} + \frac{e^{-s}(-s-1)}{s^2} + \frac{1}{s^2} + \frac{e^{-s}}{s}$$

$$\begin{matrix} t & e^{-st} \\ 1 & \downarrow \\ 0 & \downarrow \end{matrix} \quad \begin{matrix} -st \\ e^{-st} \\ -\frac{e^{-st}}{s} \\ -\frac{e^{-st}}{s^2} \end{matrix}$$



Somaiya Tabaza	الاسم	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Total
	الرقم	10	14	14	14	16	18	14	100
	المدرس								

Solve the following questions:

Q1) [10 Marks] If $W(f, g)(t) = te^{-t} - \ln t$ and if $u = f + 3g$ and $v = f - g$, find $W(u, v)(t)$.

$$W(f, g)(t) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = te^{-t} - \ln t$$

$$= fg' - gf' = te^{-t} - \ln t$$

$$W(u, v)(t) = \begin{vmatrix} f+3g & f-g \\ f'+3g' & f'-g' \end{vmatrix} = (f+3g)(f'-g') - (f-g)(f'+3g')$$

$$= \cancel{ff'} - \cancel{fg'} + \boxed{3gf'} - \cancel{3gg'} - \cancel{ff'} - \boxed{-3fg'} + \cancel{gf'} + \cancel{3gg'}$$

$$= -(fg' - gf') - 3(fg' - gf')$$

$$= -4(fg' - gf') = -4te^{-t} + 4 \ln t$$

Q2) (a) [8Marks] Find the homogeneous differential equation which has the following general solution

$$y = c_1 e^t + c_2 e^{-t} + c_3 \cos t + c_4 \sin t.$$

$$\boxed{r=1}$$

$$\boxed{r=-1}$$

$$\lambda=0, \mu=1 \quad \boxed{r=\pm i}$$

$$(r-1)(r+1)(r^2+1) = 0$$

$$(r^2-1)(r^2+1) = 0$$

$$r^4 - 1 = 0$$

b

eq. $y^{(4)} - y = 0$

(b) [6 Marks] Determine the interval in which the solution of the given initial value problem exists

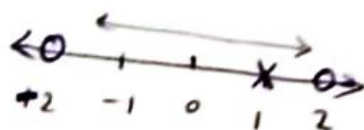
$$(4-x^2)y' + xy = -6x^2, \quad y(1) = 4.$$

$$p(x) = \frac{x}{4-x^2}$$

$$\rightarrow \text{cont. } \mathbb{R} \setminus \{\pm 2\}$$

$$g(x) = \frac{-6x^2}{4-x^2}$$

$$\rightarrow \text{cont. } \mathbb{R} \setminus \{\pm 2\}$$



$$I = (-2, 2)$$

Q3) [14 Marks] Suppose that $y_1(t) = e^t$ is one solution of the differential equation

$$ty'' - (t+1)y' + y = t^2 e^t, \quad t > 0.$$

Use the method of reduction of order to solve the differential equation.

$$y = v y = v e^t$$

$$y = v e^{t'} + v' e^t$$

$$y'' = v e^t + v' e^t + v' e^t + v'' e^t = v e^t + 2v' e^t + v'' e^t$$

$$\Rightarrow \cancel{tve^t} + 2t\underline{v'e^t} + \underline{v''e^t} - \cancel{tve^t} - \underline{t\underline{v'e^t}} - \cancel{ve^t} - \underline{v'e^t} + \cancel{ve^t} = t^2e^t$$

$$te^t v'' + (t-1)e^t v' = t^2 e^t$$

v - missing

$$v'' + \left(\frac{t-1}{t}\right)v' = t$$

$$z' + \left(1 - \frac{1}{t}\right) z = t$$

$$\begin{aligned} z &= v' \\ z' &= v'' \end{aligned}$$

By Factor method

$$\mu = e^{\int 1 - \frac{1}{t} dt} = e^t e^{-\ln t} = e^t t^{-1}$$

$$\frac{t}{z} z' + \left(\frac{t-1}{t^2}\right) e^t z = e^t$$

$$\frac{1}{2} \int \left(\frac{e^t}{t} z \right)' = \int e^t$$

$$\frac{t}{\cancel{e}} \times \frac{e^t}{T} z = e^t + C$$

$$\tau = t + \frac{t c}{c^2} \Rightarrow v' = t + \frac{t c}{c^2}$$

$$y = v e^t = \frac{t^2}{2} e^t - c_1 t - c_1 + c_2 e^t \rightarrow y$$

$$y_2 = \frac{t^2}{2} e^t - c_1(t+1)$$

$$\begin{array}{ccc} & & \int \\ & & e^{-t} \\ t & \searrow & -e^{-t} \\ & & -e^{-t} \\ & \searrow & e^{-t} \\ 6 & \longleftarrow & \end{array}$$

Q4 [14 Marks] Transform the following Euler equation into a differential equation with constant coefficients and then solve it.

$$x^2 y'' + 2xy' - 3y = \sin(\ln x), \quad x > 0.$$

$$\begin{aligned} z &= \ln x \\ e^z &= x \end{aligned}$$

$$\frac{d^2 y}{dz^2} + (2-1) \frac{dy}{dz} - 3y = \sin z$$

$$\frac{d^2 y}{dz^2} + \frac{dy}{dz} - 3y = \sin z$$

$$y = e^{rz}$$

$$\text{ch. eq. } r^2 + r - 3 = 0$$

$$r = \frac{-1 \pm \sqrt{1+4 \cdot 3}}{2}$$

$$= \frac{-1 \pm \sqrt{13}}{2}$$

$$\begin{aligned} y_c &= C_1 e^{\frac{-1-\sqrt{13}}{2} z} + C_2 e^{\frac{-1+\sqrt{13}}{2} z} \\ &= C_1 (x)^{\frac{-1-\sqrt{13}}{2}} + C_2 (x)^{\frac{-1+\sqrt{13}}{2}} \end{aligned}$$

$$\begin{aligned} y_p &= A \cos z + B \sin z \\ &= A \cos \ln x + B \sin \ln x \end{aligned}$$

$$\begin{aligned} y &= C_1 (x)^{\frac{-1-\sqrt{13}}{2}} + C_2 (x)^{\frac{-1+\sqrt{13}}{2}} + A \cos(\ln x) + B \sin(\ln x) \\ &= C_1 (x)^{\frac{-1-\sqrt{13}}{2}} + C_2 (x)^{\frac{-1+\sqrt{13}}{2}} - \frac{1}{17} \cos(\ln x) + \frac{4}{17} \sin(\ln x) \end{aligned}$$

$$\rightarrow y_p' = -A \sin z + B \cos z, \quad y_p'' = -A \cos z - B \sin z$$

$$-A \cos z - B \sin z - A \sin z + B \cos z - 3A \cos z - 3B \sin z = \sin z$$

$$[-A + B - 3A] \cos z = 0$$

$$[-B - A - 3B] \sin z = \sin z$$

$$(-4A + B = 0) \times 4$$

$$-A - 4B = 1$$

$$-17A = 1 \Rightarrow A = -\frac{1}{17}$$

$$B = \frac{-4}{17}$$

Q5 a) [10 Marks] Find and classify the singular points of the following differential equation as regular or irregular

$$(x^2 - x^3)y'' - (1+x)y' + 2xy = 0.$$

$$x^2 - x^3 = 0 \quad , \quad x^2(1-x) = 0$$

$$x=0 \quad , \quad x=1$$

$$\boxed{x=0}$$

$$\lim_{x \rightarrow 0} \frac{- (1+x)}{x^2(1-x)} = \begin{matrix} \nearrow -\infty \\ \searrow \infty \end{matrix}$$

d.n.e.

irregular

$$\boxed{x=1}$$

$$\lim_{x \rightarrow 1} \frac{- (1+x)}{x^2(1-x)} = \boxed{2}$$

$$\lim_{x \rightarrow 1} \frac{-2x}{x^2(1-x)} = 0$$

regular

(b) [6 Marks] Determine a lower bound for the radius of convergence of the series solution for the given differential equation about $x_0 = 1$

$$(x^2 - 2x - 3)y'' - 2xy' + 4y = 0.$$

$$p(x) = \frac{-2x}{x^2 - 2x - 3} = \frac{-2x}{(x-3)(x+1)}$$

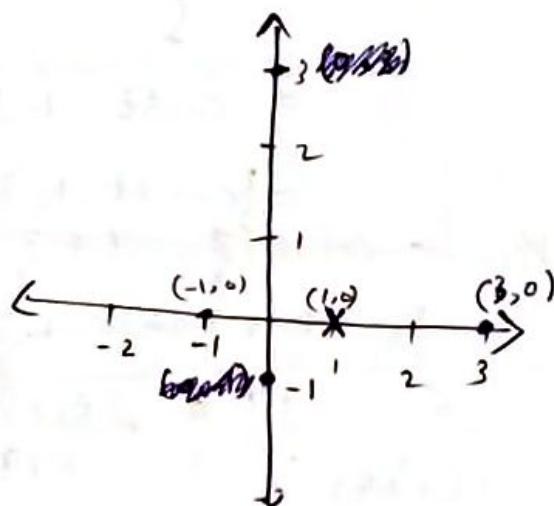
$$\mathbb{R} \setminus \{3, -1\}$$

$$q(x) = \frac{4}{x^2 - 2x - 3} \quad \mathbb{R} \setminus \{3, -1\}$$

$$L = \sqrt{(0-0)^2 + (3-1)^2} = 2$$

$$L = \sqrt{(0-0)^2 + (-1-1)^2} = \boxed{2}$$

$$\text{radius}_{\min.} = \boxed{2}$$



Q6 a) [8Marks] Use the definition to find the Laplace transformation of the function

$$f(t) = te^t, \quad t \geq 0.$$

$$L\{f(t)\} = \int_0^{\infty} e^{-st} \cdot te^t dt = \lim_{b \rightarrow \infty} \int_0^b e^{(-s+1)t} \cdot t dt$$

$$= \lim_{b \rightarrow \infty} \left[\frac{t}{(-s+1)} e^{(-s+1)t} - \frac{e^{(-s+1)t}}{(-s+1)^2} \right]_0^b$$

$$= \lim_{b \rightarrow \infty} \left[\frac{b}{-s+1} e^{(-s+1)b} - \frac{e^{(-s+1)b}}{(-s+1)^2} + \frac{1}{(-s+1)^2} \right]$$

$$= \frac{1}{(s-1)^2}$$

$\boxed{n=1}$

$$\begin{aligned} & \int_0^b e^{(-s+1)t} \cdot t dt \\ & \xrightarrow{+} \frac{e^{(-s+1)t}}{(-s+1)} \cdot t \\ & \xrightarrow{-} \frac{e^{(-s+1)t}}{(-s+1)^2} \end{aligned}$$

(b) [10 Marks] Use the Laplace transform to solve the initial value problem

$$y'' + 9y = 6, \quad y(0) = 1, \quad y'(0) = 2.$$

$$s^2 Y(s) - sy(0) - y'(0) + 9Y(s) = \frac{6}{s} \rightarrow 6\{1\} = 6 \cdot \frac{1}{s}$$

$$Y(s) [s^2 + 9] = s + 2 + \frac{6}{s}$$

$$Y(s) = \frac{s}{s^2 + 9} + \frac{2}{s^2 + 9} + \frac{6}{s(s^2 + 9)}$$

$$= \cos 3t + \frac{2}{3} \sin 3t + \frac{2}{3} \left(\frac{1}{s} \right) = \frac{2}{3} \cos 3t + \frac{2}{3} \frac{s}{s^2 + 9}$$

$$= \frac{1}{3} \cos 3t + \frac{2}{3} \sin 3t + \frac{2}{3}$$

$$Y(s) = \cos 3t + \frac{2}{3} \sin 3t + \frac{2}{3} \cos 3t + \frac{2}{3} \sin 3t$$

$$\frac{6}{s(s^2 + 9)} = \frac{A}{s} + \frac{Bs + D}{s^2 + 9}$$

$$6 = A(s^2 + 9) + s(Bs + D)$$

$$6 = As^2 + 9A + s^2B + sD$$

$$\begin{aligned} 6 &= 9A \Rightarrow A = \frac{2}{3} \\ A + B &= 0 \Rightarrow B = -\frac{2}{3} \\ D &= 0 \end{aligned}$$

$\boxed{D}$

Q7) [14 Marks] Find the first five terms in the series solution of the IVP
 $y'' + xy' + 2y = 0, y(0) = 4, y'(0) = -1.$

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + x \sum_{n=1}^{\infty} n a_n x^{n-1} + 2 \sum_{n=0}^{\infty} a_n x^n = 0$$

$n = n+2$

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n + \sum_{n=1}^{\infty} n a_n x^n + 2 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$2a_2 + 2a_0 + \sum_{n=1}^{\infty} [(n+2)(n+1)a_{n+2} + n a_n + 2a_n] x^n = 0$$

$$\boxed{a_2 = -a_0} \Rightarrow \boxed{a_2 = -4}$$

$$\boxed{n=1} \quad (6a_3 + a_1 + 2a_1)x = 0 \quad a_3 = -\frac{a_1}{2} \Rightarrow \boxed{a_3 = \frac{1}{2}}$$

$$\boxed{n=2} \quad (12a_4 + 2a_2 + 2a_2)x^2 = 0 \quad a_4 = -\frac{a_2}{3} = \frac{a_0}{3} \Rightarrow \boxed{a_4 = \frac{4}{3}}$$

$$\boxed{n=3} \quad (20a_5 + 3a_3 + 2a_3)x^3 = 0 \quad a_5 = -\frac{a_3}{4} = -\frac{a_1}{8} \Rightarrow \boxed{a_5 = -\frac{1}{8}}$$

$$y = a_0 \left[1 - \frac{x^2}{2} + \frac{x^4}{3} - \dots \right] + a_1 \left[x - \frac{x^3}{2} + \frac{x^5}{8} - \dots \right]$$

$$y = a_0 \left[1 - \frac{x^2}{2} + \frac{x^4}{3} - \dots \right] + a_1 \left[x - \frac{x^3}{2} + \frac{x^5}{8} - \dots \right]$$

$$y(0) = 4 \Rightarrow \boxed{a_0 = 4}$$

$$y' = a_0 \left[-2x + \frac{4}{3}x^3 - \dots \right] + a_1 \left[1 - \frac{3}{2}x^2 + \frac{5}{8}x^4 - \dots \right]$$

$$y'(0) = -1 = \boxed{a_1}$$

Good Luck

Q.4 Use the Laplace transform method to solve the initial value problem

$$y''' - y'' = \sin(2t), \quad y(0) = y'(0) = y''(0) = 3$$

$$\underline{s^3 Y(s)} - s^2 y(0) - s y'(0) - y''(0) - s^2 Y(s) + s y(0) + y'(0) = L\{\sin 2t\}$$

$$= \frac{2}{s^2 + 4}$$

$$Y(s) [s^2(s-1)] - 3s^2 - \cancel{3s} - \cancel{3} + \cancel{3s} + 3 = \frac{2}{s^2 + 4}$$

$$Y(s) = \frac{2}{s^2(s-1)(s^2+4)} + \frac{\cancel{3s^2}}{\cancel{s^2}(s-1)}$$

$$= \frac{A}{s} + \frac{-\frac{1}{2}}{s^2} + \frac{\frac{2}{5}}{(s-1)} + \frac{Ds+E}{s^2+4} \quad \left(\frac{Ds}{s^2+4} + \frac{E}{s^2+4} \right) + \frac{3}{s-1}$$

$$= A + \frac{-\frac{1}{2}}{s} + \frac{2}{5} e^t + D \cos 2t + \frac{E}{2} \sin 2t + 3e^t$$

$$= A + \frac{-\frac{1}{2}}{s} + \frac{17}{5} e^t + D \cos 2t + \frac{E}{2} \sin 2t$$

$$\frac{2}{s^2(s-1)(s^2+4)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{(s-1)} + \frac{Ds+E}{s^2+4}$$

$$2 = A s(s-1)(s^2+4) + B(s-1)(s^2+4) + C s^2(s^2+4) + (Ds+E)s^2(s-1)$$

$$\boxed{s=0} \quad 2 = -4B \Rightarrow \boxed{B = -\frac{1}{2}}$$

$$\boxed{s=1} \quad 2 = 5C \Rightarrow \boxed{C = \frac{2}{5}}$$

1
1
1
1