

Time Domain Analysis of continuous time system

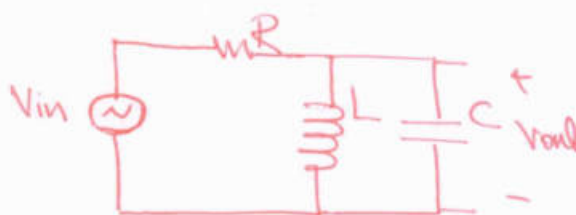
→ if you want to study and analysis any system you need to perform these steps:

1) Make modeling for a system like RLC circuit

2) Develop mathematical equations

3) Analysis.

4) Design the new system



Modeling means:- it is a translator that represent a physical device by its equivalent equation.

— which depend on your analysis types:

→ In our course we have two types for analysis a linear time invariant system which are:

1) Time domain method

a) Differential equation
(Derivatives)

b) Convolution
(Integral)

2) Frequency domain method

a) Transfer function
(Laplace transform)

b) Frequency Response
(Integral Fourier Transform)

(2)

→ In this chapter we will discuss the type one of analysis for continuous time system.

→ as we said before we can say the system is continuous time iff the variables are continuous.

$$y(t) = x(t) \rightarrow t \text{ is continuous.}$$

→ there are standard form for differential equation to LTI system:

$$a_n \frac{d^n y(t)}{dt^n} + a_{n-1} \frac{d^{n-1} y(t)}{dt^{n-1}} + \dots + a_0 y(t) = b_m \frac{d^m x(t)}{dt^m} + \dots + b_0 x(t)$$

where a_i & b_i are constant & $N \geq M$

Ex2

$$a_3 \frac{d^3 y(t)}{dt^3} + a_2 \frac{d^2 y(t)}{dt^2} + a_1 \frac{dy}{dt} + a_0 y(t) = b_3 \frac{d^3 x(t)}{dt^3} + b_2 \frac{d^2 x(t)}{dt^2} + b_1 \frac{dx(t)}{dt} + b_0$$

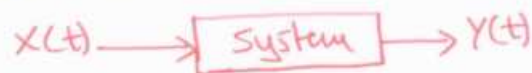
$M=3$ & $N=3$ 3rd order system

To solve a differential equation you need to find $y(t)$ by substitute of $x(t)$ (given)

→ The response of linear system can be express as the sum of two component:

- 1) zero-input response (called natural response)
- 2) zero-state response (called forced response)

→ The two component are independent of each other 3
 each one can be computed independently of the other.



Total response = zero input response + zero state response

- when $X(t) = \text{zero}$
- result from internal system condition
(energy storage, Initial condition)
- response to external input $X(t)$ when the system condition are zero

e.g

$$y(t) = y(t_0) e^{-\frac{1}{RC}(t-t_0)} + \int_{t_0}^t \frac{1}{C} e^{-\frac{1}{RC}(t-\lambda)} x(\lambda) d\lambda$$

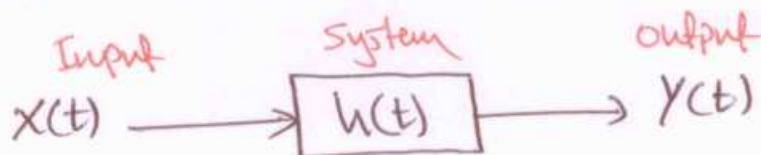
↳ this part terminate when $t \rightarrow \infty$

↳ this part terminate when $X(t) = \text{zero}$. no input.

→ So, In this course we will focus on finding the zero-state response ($I.C = \text{zero}$)

⊗ Convolution (Integral) :-

- It is an operation give the output of a system $h(t)$ if we applied the input $x(t)$.



$h(t)$:- its the Impulse response of system

⊗ equations:-

[4]

$$y(t) = \int_{-\infty}^{\infty} x(\tau) \cdot h(t-\tau) d\tau$$

OR

$$y(t) = \int_{-\infty}^{\infty} x(t-\tau) \cdot h(\tau) d\tau$$

$$= x(t) * h(t)$$

↑

convolution

→ There are two methods to apply convolution:-

1) mathematical. 2) graphical.

① Mathematical:-

→ There are some properties of the convolution:-

1) Commutative:- संज्ञा

$$x(t) * h(t) = h(t) * x(t)$$

2) Associative:- (series connection)

$$x(t) * h_1(t) * h_2(t) = [x(t) * h_1(t)] * h_2(t) = [x(t) * h_2(t)] * h_1(t)$$

3) Distributive:- (parallel connection)

$$x(t) * [h_1(t) + h_2(t)] = [x(t) * h_1(t)] + [x(t) * h_2(t)]$$

4) Convolution with an impulse:-

$$x(t) * \delta(t) = x(t)$$

5) Convolution with unit step:-

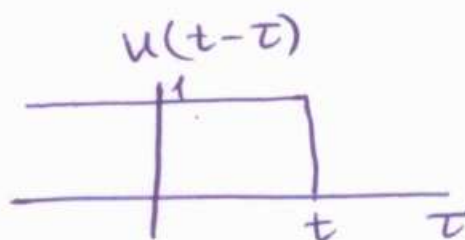
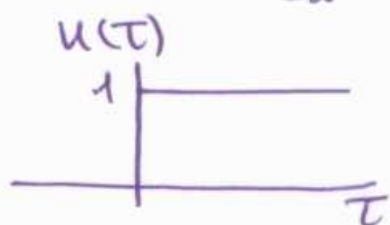
$$x(t) * u(t) = \int_{-\infty}^{\infty} x(\tau) u(t-\tau) d\tau = \int_{-\infty}^t x(\tau) d\tau$$

Ex 1 Find the convolution for $u(t) * u(t)$

[5]

By rule-

$$u(t) * u(t) = \int_{-\infty}^{\infty} u(\tau) u(t-\tau) d\tau$$



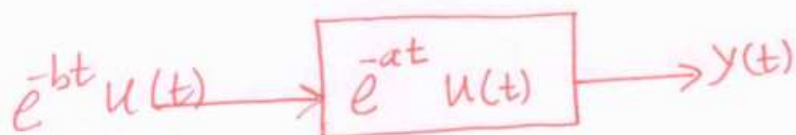
$$= \int_{-\infty}^{\infty} u(\tau) u(t-\tau) d\tau$$

$$= \int_0^t 1 d\tau = \tau \Big|_0^t = \boxed{t} \quad \text{ramp signal.}$$

OK By properties-

$$u(t) * u(t) = \int_{-\infty}^{\infty} u(\tau) u(t-\tau) d\tau$$
$$= \int_{-\infty}^t u(t) = r(t) \quad \text{ramp signal.}$$

Ex 2: Find $y(t)$?



$a > 0$
 $b \neq a$

$$y(t) = u(t) * x(t)$$

$$= \int_{-\infty}^{\infty} e^{-b\tau} u(\tau) \cdot e^{-a(t-\tau)} u(t-\tau) d\tau$$

$$u(\tau) \cdot u(t-\tau) = 1 \quad 0 \leq \tau \leq t \quad (6)$$

$$y(t) = \int_0^t e^{-b\tau} \cdot e^{-a(t-\tau)} d\tau$$

$$= \int_0^t \underbrace{e^{-at}}_{\text{constant}} \cdot e^{(a-b)\tau} d\tau$$

$$= e^{-at} \int_0^t e^{(a-b)\tau} d\tau = e^{-at} \left. \frac{e^{(a-b)\tau}}{(a-b)} \right|_0^t$$

$$= e^{-at} \cdot \left(\frac{e^{(a-b)t}}{(a-b)} - \frac{1}{(a-b)} \right)$$

$$y(t) = \left[\frac{e^{-bt}}{a-b} - \frac{e^{-at}}{a-b} \right] \underline{\underline{u(t)}}$$

Do not forget it
to be causal
system

Ex 3

$$\text{if } x(t) = 10e^{-3t} u(t) \quad \& \quad h(t) = [2e^{-2t} - e^{-t}] u(t)$$

Find y(t)?

$$y(t) = x(t) * h(t)$$

$$= \int_{-\infty}^{\infty} 10e^{-3\tau} u(\tau) \cdot 2e^{-2(t-\tau)} u(t-\tau) d\tau -$$

$$\int_{-\infty}^{\infty} 10e^{-3\tau} u(\tau) \cdot e^{-(t-\tau)} u(t-\tau) d\tau$$

$$\begin{aligned}
&= \int_0^t 10 e^{-3\tau} \cdot 2 e^{-2t} \cdot e^{2\tau} d\tau - \int_0^t 10 e^{-3\tau} \cdot e^{-t} \cdot e^{\tau} d\tau \quad \boxed{7} \\
&= 20 e^{-2t} \int_0^t 10 e^{-\tau} d\tau - 10 e^{-t} \int_0^t e^{-2\tau} d\tau \\
&= 20 e^{-2t} \cdot \left. \frac{e^{-\tau}}{-1} \right|_0^t - 10 e^{-t} \cdot \left. \frac{e^{-2\tau}}{-2} \right|_0^t \\
&= 20 e^{-2t} \cdot [-e^{-t} + 1] - \frac{10 e^{-t}}{2} [e^{-2t} - 1] \\
&= -20 e^{-3t} + 20 e^{-2t} - 5 e^{-3t} + 5 e^{-t} \quad u(t) \\
&= [20 e^{-2t} - 15 e^{-3t} + 5 e^{-t}] u(t) \quad \mathcal{R}
\end{aligned}$$

Notes: See page 177 in text book to see convolution table.

Ex 24 $x(t) = t e^{-t} u(t)$, $h(t) = u(t)$ Find $y(t)$

$$y(t) = \int_{-\infty}^{\infty} \tau e^{-\tau} u(\tau) \cdot \underbrace{u(t-\tau)}_{=1} d\tau$$

$$= \int_{-\infty}^t \tau e^{-\tau} d\tau$$

$$= \int_0^t -\tau e^{-\tau} d\tau = -\tau e^{-\tau} - e^{-\tau} \Big|_0^t$$

$$= [-t e^{-t} - e^{-t} + 1] u(t)$$

$\mathcal{R}$

$$\begin{array}{rcl}
\tau & \cdot & e^{-\tau} \\
& \searrow + & \\
1 & \cdot & -e^{-\tau} \\
& \searrow - & \\
0 & \cdot & e^{-\tau} \\
& \searrow - & \\
& & -e^{-\tau}
\end{array}$$

$$= -\tau e^{-\tau} - e^{-\tau}$$

Ex5 if $x(t) = u(t)$, $h(t) = e^{-2t} u(t) + \delta(t)$

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Find the $y_1(t)$ & $y_2(t)$ & $y_T(t)$

$$- y_1 = \int_{-\infty}^{\infty} u(t-\tau) \cdot e^{-2\tau} u(\tau) d\tau = \int_0^t e^{-2\tau} d\tau$$

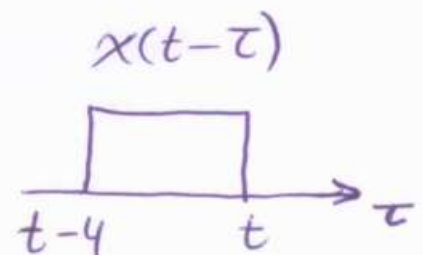
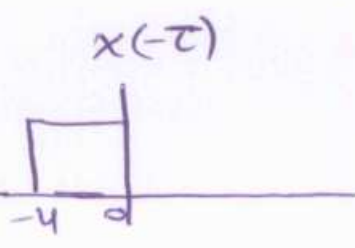
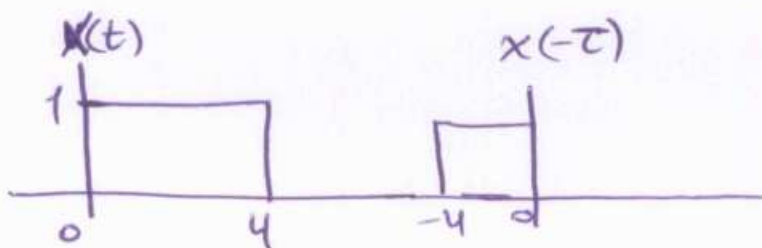
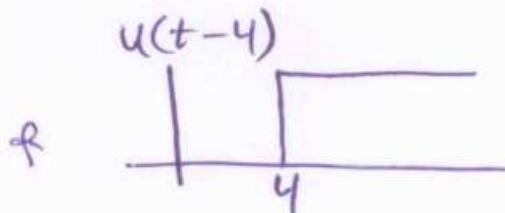
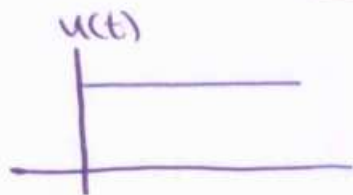
$$= \left. \frac{e^{-2\tau}}{-2} \right|_0^t = \left[\frac{e^{-2t}}{-2} + \frac{1}{2} \right] u(t)$$

$$- y_2 = \int_{-\infty}^{\infty} u(t-\tau) \cdot \delta(\tau) d\tau = u(t)$$

$$- y_T = \left[\frac{e^{-2t}}{-2} + \frac{1}{2} \right] u(t) + u(t) \quad \text{A.}$$

VIP A linear time invariant system has the following impulse response $h(t) = 2e^{-at} u(t)$ use convolution to find the response $y(t)$ to the following input

$x(t) = u(t) - u(t-4)$ & sketch $y(t)$ for the case when $a=1$



$\therefore y(t)$ has 3 case 1) $t < 0$ 2) $0 \leq t < 4$ 3) $t \geq 4$ (9)

at $t < 0$ $y(t) = 0$

at $0 \leq t < 4$ $y(t) = \int_0^t 2e^{-a\tau} d\tau = \frac{2e^{-a\tau}}{-a} \Big|_0^t$

$$y(t) = \frac{2}{-a} (e^{-at} - 1)$$

at $t \geq 4$ $y(t) = \int_{t-4}^t 2e^{-a\tau} d\tau = \frac{2}{-a} e^{-a\tau} \Big|_{t-4}^t$

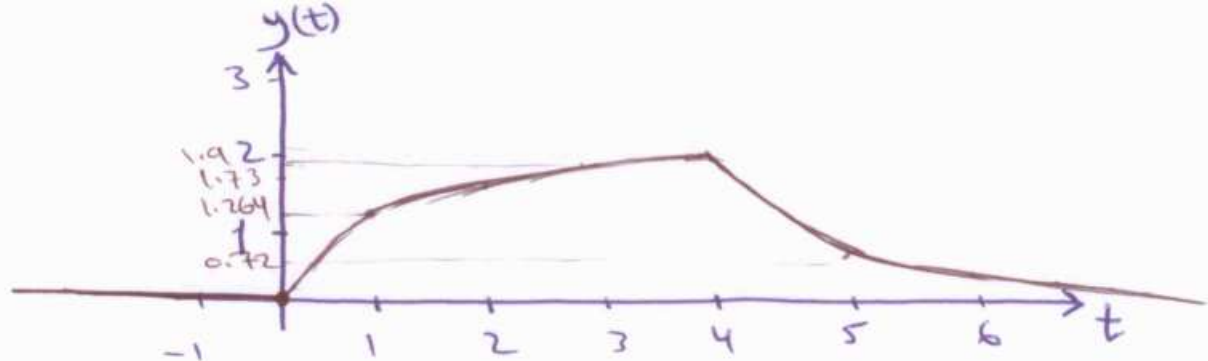
$$y(t) = \frac{2}{-a} [e^{-at} - e^{-a(t-4)}]$$

$$\therefore y(t) = \begin{cases} 0 & t < 0 \\ -\frac{2}{a}(e^{-at} - 1) & 0 \leq t < 4 \\ -\frac{2}{a}(e^{-at} - e^{-a(t-4)}) & t \geq 4 \end{cases}$$

To Be
Causal
system

at $a=1$

$$y(t) = \begin{cases} 0 & t < 0 \\ -2(e^{-t} - 1) & 0 \leq t < 4 \\ -2(e^{-t} - e^{-(t-4)}) & t \geq 4 \end{cases}$$



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[2] Graphical Convolution Operation :-

III

→ Many signals have no exact mathematical description
So they can be described only graphically.

→ The procedures :-

(^{here} ~~old~~ ~~loop~~ ~~to~~ ~~be~~ ~~is~~ ~~also~~ ~~also~~)

1) Convert the variable t by τ in $x(t) \rightarrow x(\tau)$
 $h(t) \rightarrow h(\tau)$

2) Keep one signal fixed like $x(\tau)$ or $h(\tau)$

then find the inversion of the another one $h(\tau) \rightarrow h(-\tau)$
or $x(\tau) \rightarrow x(-\tau)$

3) Shift the inverted signal by t $h(-\tau) \rightarrow h(t-\tau)$
or $x(-\tau) \rightarrow x(t-\tau)$

if t is positive $\Rightarrow$ (right shift)

if t is negative $\Rightarrow$ (left shift)

4) Now, determine the critical points or change points

* Divide the two signals into regions depends on this point.

(change in Amp or slope) To determine the integration bounds

5) The area under the product $x(\tau) \cdot h(t-\tau)$ is the value of convolution at t

6) Do the integral $y(t) = \int_{-\infty}^{\infty} x(\tau) \cdot h(t-\tau) d\tau$

Notes 2 What is the rectangular pulse

→ it is a linear combination from two step function
To build a window called rectangular pulse

→ equal

$$X(t) = A \operatorname{rect} \left(\frac{t - n}{m} \right)$$

where

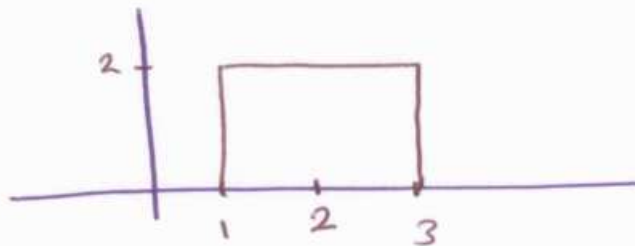
A : amplitude

n : center of ~~rect~~ ^{rect}

m : the length of rect

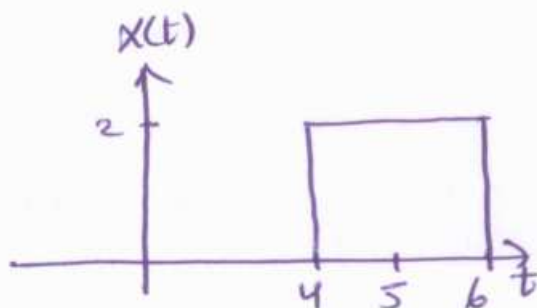
Example 1

$$X(t) = 2 \operatorname{rect} \left(\frac{t - 2}{2} \right)$$

Example 2

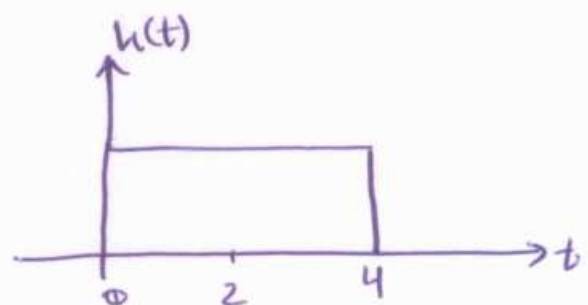
$$X(t) = 2 \operatorname{rect} \left(\frac{t - 5}{2} \right)$$

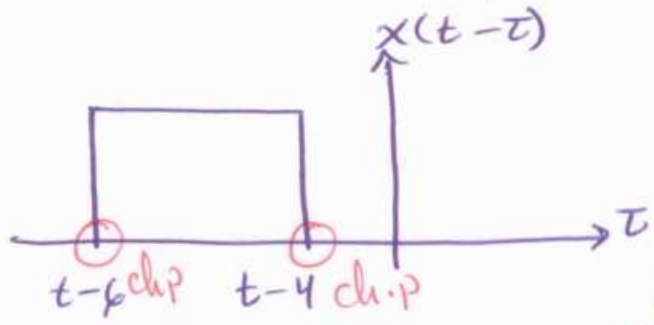
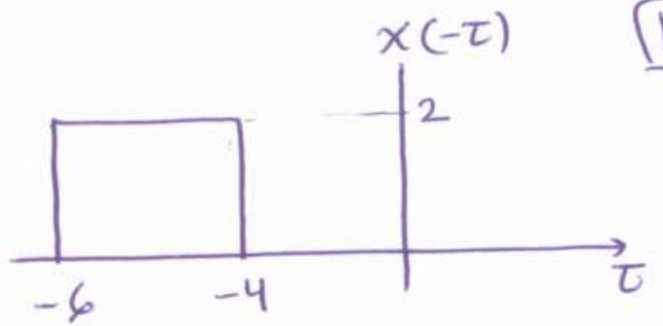
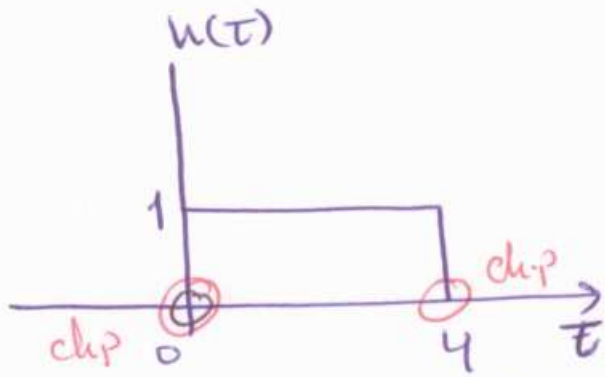
$$h(t) = \operatorname{rect} \left(\frac{t - 2}{4} \right)$$



$$X(t) \rightarrow \boxed{h(t)} \rightarrow Y(t)$$

Sketch $Y(t)$?





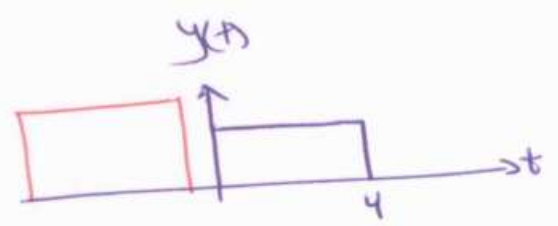
* حدد نقاط التكول على التناظر

رتب التناظر في التناظر

$$\left. \begin{aligned}
 t-4 = 0 &\Rightarrow t=4 \\
 t-4 = 4 &\Rightarrow t=8 \\
 t-6 = 0 &\Rightarrow t=6 \\
 t-6 = 4 &\Rightarrow t=10
 \end{aligned} \right\} \Rightarrow y(t) = \begin{cases} 0 & t < 4 \\ 2t-8 & 4 \leq t < 6 \\ 4 & 6 \leq t < 8 \\ 20-2t & 8 \leq t < 10 \\ 0 & t \geq 10 \end{cases}$$

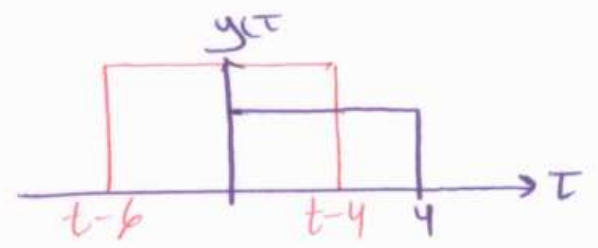
The output 4 → 10

1) at $t < 4$
 $y = 0$



2) at $4 \leq t < 6$

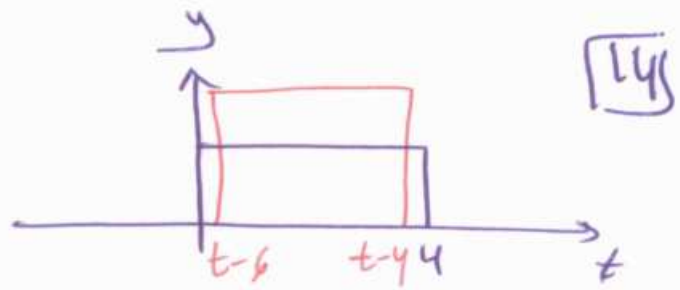
$$y(t) = \int_0^{t-4} (2)(1) d\tau = 2\tau \Big|_0^{t-4} = 2t-8$$



3) at $6 \leq t < 8$

$$y_2 = \int_{t-6}^{t-4} (2)(1) d\tau = 2\tau \Big|_{t-6}^{t-4}$$

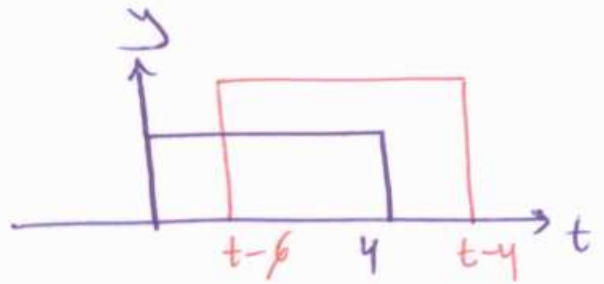
$$= 2t - 8 - (2t - 12) = 4$$



4) at $8 \leq t < 10$

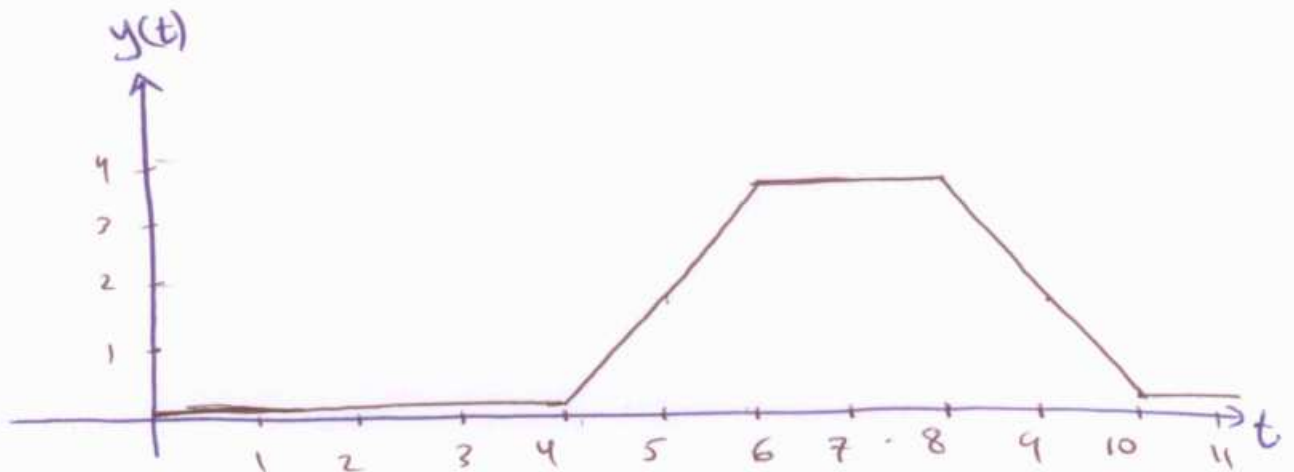
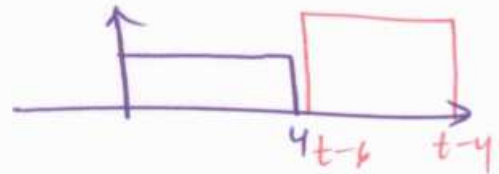
$$y_2 = \int_{t-6}^4 2 d\tau = 2\tau \Big|_{t-6}^4$$

$$= 8 - 2t + 12 = 20 - 2t$$



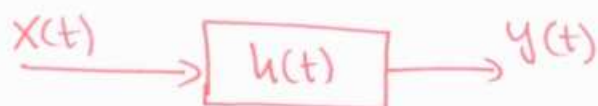
5) at $t \geq 10$

$$y_2 = 0$$

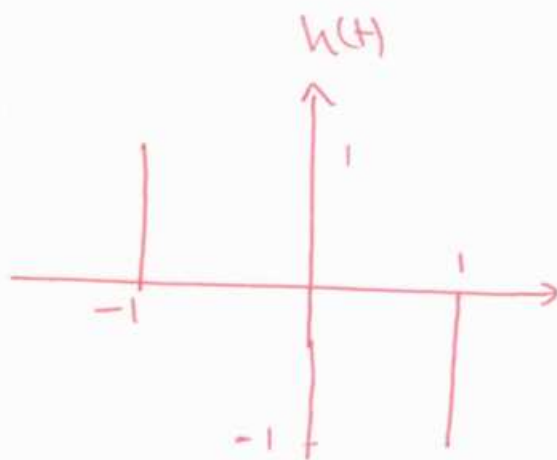
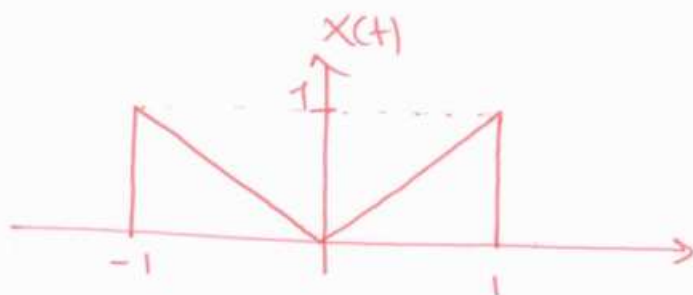


Example 2

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Find $y(t)$ when



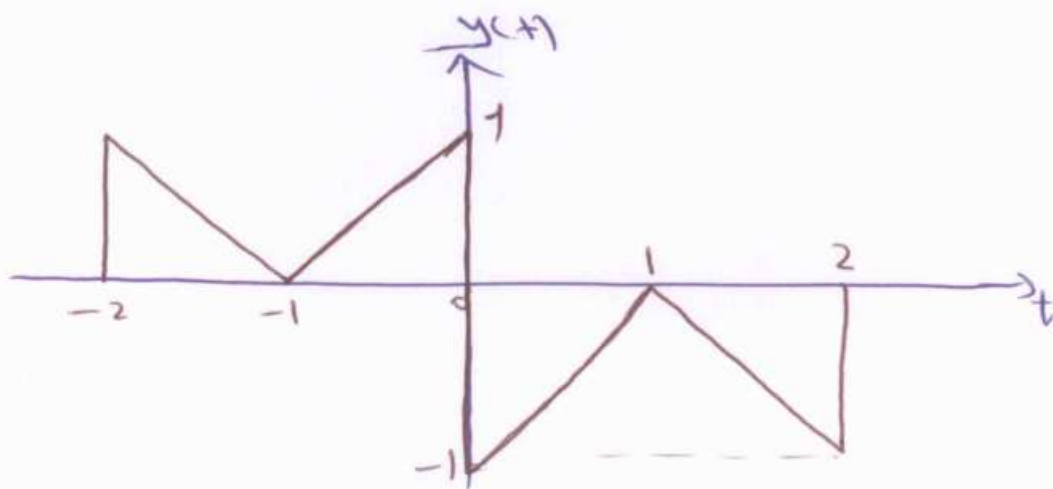
By using the properties

$$y(t) = x(t) * h(t) = x(t) * \delta(t) = x(t)$$

$$h(t) = \delta(t+1) - \delta(t-1)$$

$$\therefore y(t) = x(t) * [\delta(t+1) - \delta(t-1)]$$

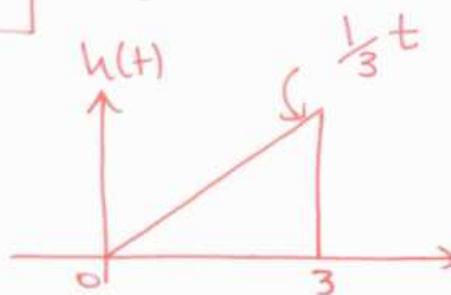
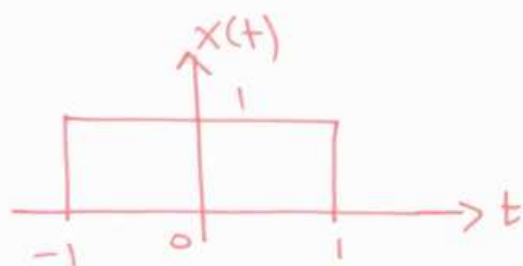
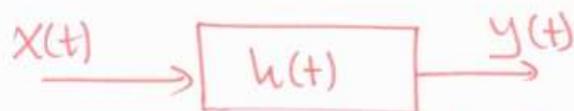
$$y(t) = x(t+1) - x(t-1)$$



H.

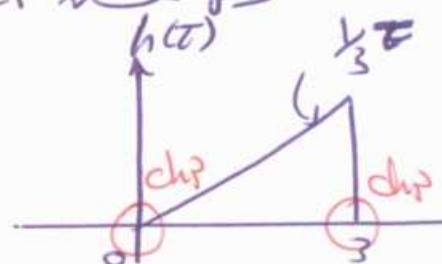
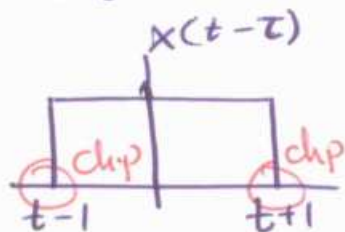
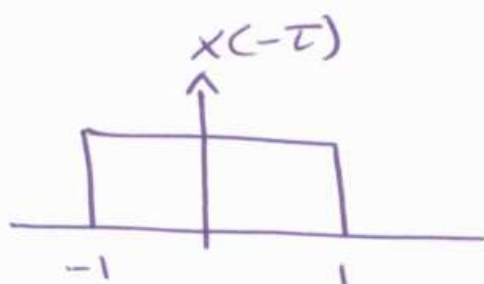
Example

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sketch $y(t)$

ملاحظة: shift, Reflection, length.
 الانكسار في الترددات العالية.
 الانكسار في الترددات العالية.



$$t+1 \geq 0 \Rightarrow t \geq -1$$

$$t+1 \geq 3 \Rightarrow t \geq 2$$

$$t-1 \geq 0 \Rightarrow t \geq 1$$

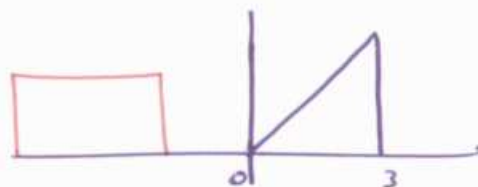
$$t-1 \geq 3 \Rightarrow t \geq 4$$

$$y(t) = \begin{cases} 0 & t \leq -1 \\ \frac{1}{6}(t^2 + 2t + 1) & -1 \leq t < 1 \\ \frac{2}{3}t & 1 \leq t < 2 \\ \frac{3}{2} - \frac{1}{6}(t^2 + 2t + 1) & 2 \leq t < 4 \\ 0 & t \geq 4 \end{cases}$$

output rang -1 to 4

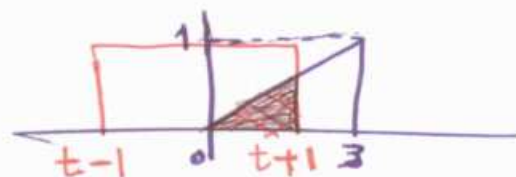
1) at $t \leq -1$

$$y = 0$$



2) at $-1 \leq t < 1$

$$y = \int_{-1}^{t+1} \frac{1}{3}\tau d\tau = \frac{1}{6}\tau^2 \Big|_{-1}^{t+1}$$



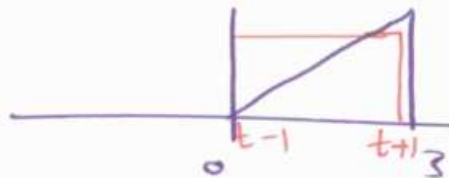
$$= \frac{1}{6}((t+1)^2) = \frac{1}{6}(t^2 + 2t + 1)$$

3) ab $1.5t < 2$

$$y = \int_{t-1}^{t+1} \frac{1}{3} \tau d\tau = \frac{1}{6} \tau^2 \Big|_{t-1}^{t+1}$$

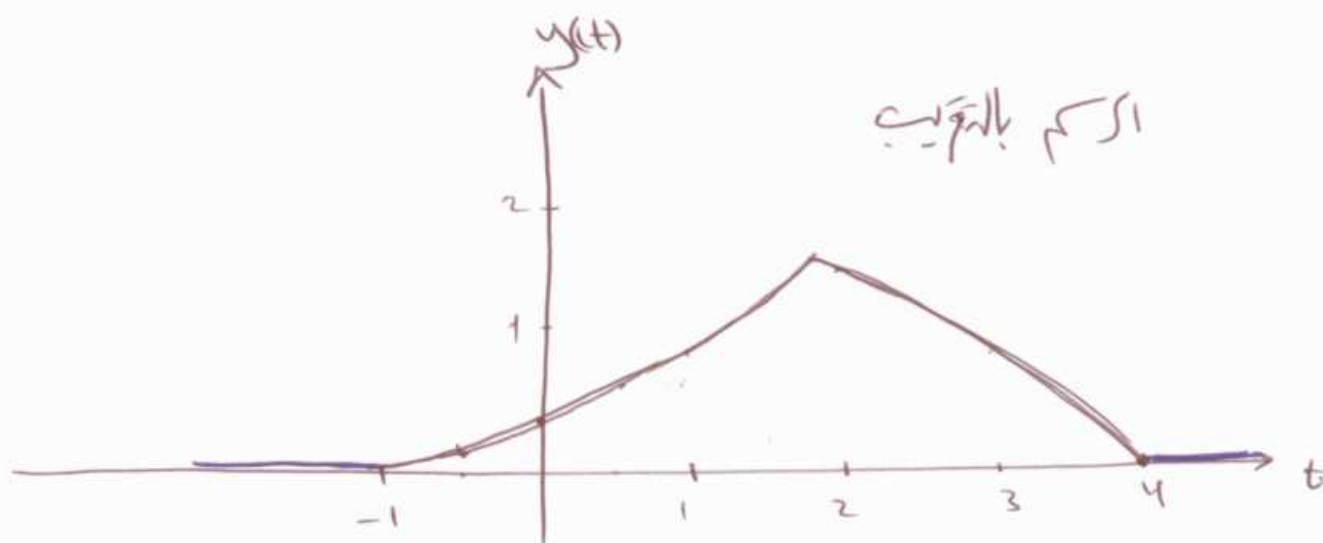
$$= \frac{1}{6} ((t^2 + 2t + 1) - (t^2 - 2t + 1))$$

$$= \frac{4t}{6} = \frac{2}{3}t$$

4) ab $2 \leq t < 4$

$$y = \int_{t-1}^3 \frac{1}{3} \tau d\tau = \frac{1}{6} \tau^2 \Big|_{t-1}^3 = \frac{1}{6} (9 - (t^2 - 2t + 1))$$

$$= \frac{3}{2} - \frac{1}{6} (t^2 - 2t + 1)$$



اسم بالقرآن

⊗ System Stability-

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- The system is stable if it results in a bounded output for any bounded input. (BIBO)
- A system is said to be BIBO stable if for any bounded input $|x(t)| \leq C_1 < \infty$ the output remains bounded $|y(t)| \leq C_2 < \infty$

∴ Stability: CT LTI system is stable if

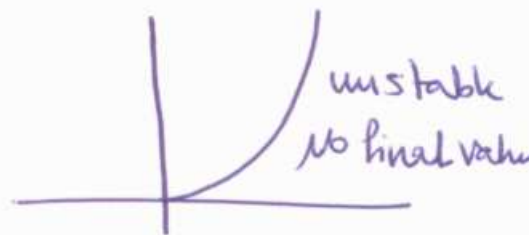
$$\int_{-\infty}^{\infty} h(\tau) d\tau < \infty$$

Example 8-

$$h_1(t) = -3e^{2t} u(t)$$

$$\int_{-\infty}^{\infty} |h(t)| d\tau = \int_0^{\infty} -3e^{2t} = \infty$$

unstable



$$h_2(t) = e^{4t} u(-t)$$

$$\int_{-\infty}^0 e^{4t} = 1 - 0 = 1 \quad \text{stable}$$

