

Chapter 4Laplace transform (LT)

- As we studied in chapter 7 (Fourier transform) for continuous signals and saw how it enable us to do a lot of things (its Applications).

⇒ So why do we need yet another transform?

- The Laplace transform is an extension of the Fourier transform to allow analysis of broader classes of signals and systems.
- But Fourier transform cannot handle large classes of signal and unstable systems.

The condition of existence Fourier transform is 2

$$\int_{-\infty}^{\infty} |x(t)| dt < \infty \quad \text{the integral must be convergent}$$

So, if the integral is divergence the Fourier transform of the signal does not exist.

⇒ Now, by using Laplace transform we can find the frequency representation for this type of signal, because there is no conditions or limitations for using this type of transform.

— Also, the Laplace transform transforms the problem of D.eq from time domain to the frequency domain then transform the solution of D.eq from frequency domain to time domain by using inverse transform

* Laplace transform 2

$$X(s) = \mathcal{L}\{x(t)\} = \int_{-\infty}^{\infty} x(t) \cdot e^{-st} dt \quad \left(\begin{array}{l} \text{for causal} \\ \text{non causal signals} \end{array} \right)$$

Bilateral (two-sided Laplace transform)

$$X(s) = \mathcal{L}\{x(t)\} = \int_0^{\infty} x(t) \cdot e^{-st} dt \quad \left(\begin{array}{l} \text{Just for causal} \\ \text{signals} \end{array} \right)$$

where $s = \sigma + j\omega$

complex number real part Imaginary part

Ex 1 $x(t) = e^{\alpha t} u(t)$, find the frequency domain?

By FT $\int_{-\infty}^{\infty} |e^{\alpha t}| dt = \int_0^{\infty} e^{\alpha t} dt = \infty$ (div.) FT is not exist

By LT $\int_{-\infty}^{\infty} |x(t)| dt = \int_0^{\infty} e^{\alpha t} dt = \infty$ LT exist so

$$X(s) = \int_{-\infty}^{\infty} e^{\alpha t} \cdot e^{-st} dt = \int_0^{\infty} e^{(\alpha-s)t} dt = \frac{-1}{\alpha-s} = \boxed{\frac{1}{s-\alpha}}$$

$$\text{ROC} \Rightarrow \text{Re}\{s\} = \sigma > \alpha$$

• What is ROC?

↳ ROC means (region of convergence) or its called the region of existence

$$\text{if } \int_0^{\infty} e^{\alpha t} \cdot e^{-st} dt = \int_0^{\infty} e^{\alpha t} \cdot e^{-(\sigma + j\omega)t} dt$$

$$= \int_0^{\infty} e^{-(\alpha + \sigma)t} \cdot e^{-j\omega t} dt$$

To be this integral convergence the value of $e^{-(\alpha + \sigma)t}$ must be convergence (how)!

By $(-\alpha + \sigma)$ be greater than zero (positive value).

$$-\alpha + \sigma > 0 \Rightarrow \boxed{\sigma > \alpha}$$

↳ And this ROC is needed to compute the inverse Laplace transform (using table).

Ex² $x(t) = e^{-bt} u(t)$ find $X(s)$

$$X(s) = \int_0^{\infty} e^{-bt} \cdot e^{-st} dt = \int_0^{\infty} e^{-(s+b)t} dt = \frac{1}{s+b}$$

$$\text{ROC} \Rightarrow \sigma + b > 0 \Rightarrow \sigma > -b \Rightarrow \boxed{\text{Re}\{s\} > -b}$$

Ex2 $x(t) = -e^{-bt} u(-t)$, find $X(s)$

$$X(s) = \int_{-\infty}^0 -e^{-bt} \cdot e^{-st} dt = \int_{-\infty}^0 -e^{-(b+s)t} dt$$

$$X(s) = \frac{1}{s+b}$$

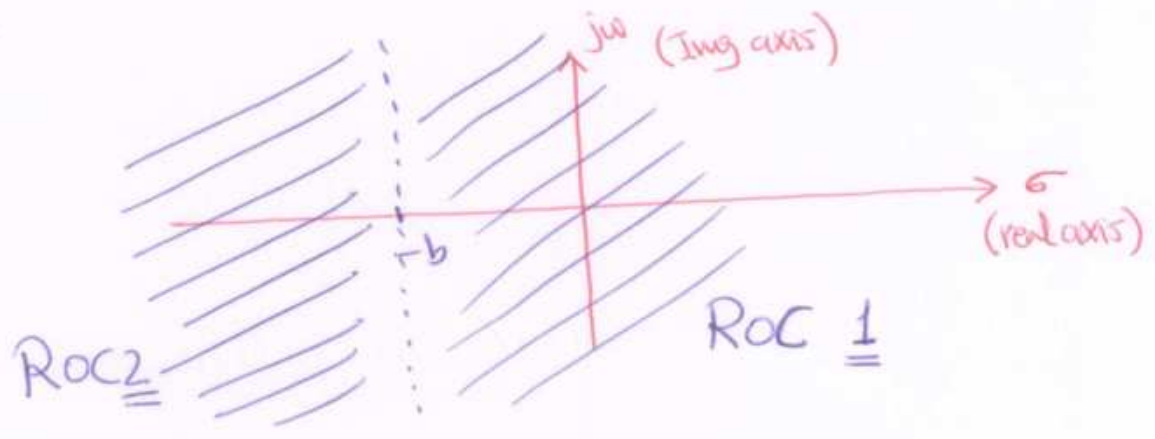
للجواب

$$\text{Roc} \Rightarrow \sigma + b < 0 \Rightarrow \sigma < -b \Rightarrow \text{Roc } \text{Re}\{s\} < -b$$

لأننا معرفة $x(t) \rightarrow 0$ يعني $t \rightarrow \infty$

هنا المثل السابق لم نجد الجواب $X(s)$ ، فإذا طلب منا استرجاع $x(t)$ من $X(s)$ دون تحديد أي من المثلين هو المقصود لذلك يجب أن نكتب Roc في سطح استرجاع الإشارة .

⊗ S-plane 2



Ex: $x(t) = 3e^{-2t} u(t) + 4e^t u(-t)$ Find $X(s)$

→ you can get the answer from the table.

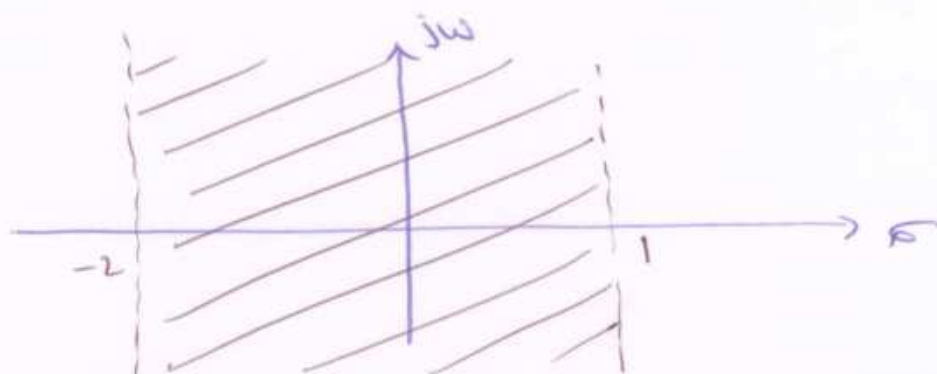
$$\begin{aligned} X(s) &= \int_0^{\infty} 3e^{-2t} \cdot e^{-st} dt + \int_{-\infty}^0 4e^t \cdot e^{-st} dt \\ &= 3 \int_0^{\infty} e^{-(s+2)t} dt + 4 \int_{-\infty}^0 e^{-(s-1)t} dt \end{aligned}$$

$$X(s) = \frac{3 e^{-(s+2)t}}{-(s+2)} \Big|_0^\infty + \frac{4 e^{-(s-1)t}}{-(s-1)} \Big|_0^\infty$$

$$X(s) = \frac{3}{s+2} - \frac{4}{s-1}$$

$$\text{Roc } \text{Re}\{s\} > -2 \quad \bigcap \quad \text{Roc } \text{Re}\{s\} < 1$$

$$\text{Roc } -2 < \text{Re}\{s\} < 1$$



⊛ Find the Laplace transform for the following signals:

① $x(t) = \delta(t)$

$$X(s) = \int_0^\infty \delta(t) e^{-st} dt = e^0 = \underline{1}$$

$$\text{Roc} \Rightarrow \text{Re}\{s\} > 0$$

② $x(t) = u(t)$

$$X(s) = \int_0^\infty e^{-st} dt = \frac{e^{-st}}{-s} \Big|_0^\infty = \frac{1}{s}$$

$$\text{Roc} \Rightarrow \text{Re}\{s\} > 0$$

③ $x(t) = e^{jat} u(t)$

$$X(s) = \int_0^{\infty} e^{jat} \cdot e^{-st} dt$$

$$= \int_0^{\infty} e^{-(s-j a)t} dt = \frac{1}{s-j a} \cdot \frac{s+j a}{s+j a}$$

$$\text{Re}\{s\} > 0$$

نوع الدالة هو الدالة العظمى
لذلك لا يوجد شرط على $\text{Im}\{s\}$

$$X(s) = \frac{s+j a}{s^2 + a^2}$$

$$\text{Roc} \Rightarrow \text{Re}\{s\} > 0$$

④ $x(t) = \cos \omega_0 t u(t)$

→ دالة

Notes

$$e^{j\omega_0 t} = \cos \omega_0 t + j \sin \omega_0 t$$

$$\mathcal{L}\{e^{j\omega_0 t}\} = \frac{s}{s^2 + \omega_0^2} + j \frac{\omega_0}{s^2 + \omega_0^2}$$

$\Downarrow$ $\mathcal{L}\{\cos \omega_0 t\}$ $\Downarrow$ $\mathcal{L}\{\sin \omega_0 t\}$

$$\therefore X(s) = \frac{s}{s^2 + \omega_0^2}$$

$$\text{Roc} \Rightarrow \underline{\text{Re}\{s\} > 0}$$

⑤ $x(t) = \sin \omega_0 t u(t)$

$$X(s) = \frac{\omega_0}{s^2 + \omega_0^2}$$

$$\text{Roc} \Rightarrow \underline{\text{Re}\{s\} > 0}$$

⊛ properties of Laplace transform 2-

[1] linearity 2

$$x_1(t) \longleftrightarrow X_1(s)$$

$$x_2(t) \longleftrightarrow X_2(s)$$

$$a x_1(t) + b x_2(t) \longleftrightarrow a X_1(s) + b X_2(s)$$

[2] Time shifting 2

$$x(t) \longleftrightarrow X(s)$$

(shift in time)

$$x(t - t_0) \longleftrightarrow e^{-s t_0} X(s)$$

تحويل في الزمن

(shift in frequency)

$$e^{s_0 t} \cdot x(t) \longleftrightarrow X(s - s_0)$$

تحويل في التردد

[3] Time scaling 2

$$x(t) \longleftrightarrow X(s)$$

$$x(at) \longleftrightarrow \frac{1}{a} X\left(\frac{s}{a}\right)$$

[4] Differentiator & integrator in Time domain 2

$$x(t) \longleftrightarrow X(s)$$

T.F. في التردد
D. eq. في الزمن

$$\frac{d^n x(t)}{dt^n} \longleftrightarrow s^n X(s) - s^{n-1} X(0) - s^{n-2} X'(0) - \dots - X^{(n-1)}(0)$$

$$\int_0^t x(\tau) d\tau \longleftrightarrow \frac{1}{s} X(s)$$

Ex 1

18

$$y''(t) + 3y'(t) + 2y(t) = 0 \quad , \quad y(0) = 3, \quad y'(0) = 1$$

Find $y(t)$

↳ convert to s -domain

$$[s^2 Y(s) - sY(0) - Y'(0)] + 3[sY(s) - Y(0)] + 2Y(s) = 0$$

$$Y(s)(s^2 + 3s + 2) - 3s - 1 - 9 = 0$$

$$Y(s) = \frac{3s + 10}{s^2 + 3s + 2}$$

↳ now, find the inverse Laplace transform

5 Differentiation in s -domain

$$x(t) \longleftrightarrow X(s)$$

$$(-t)^n x(t) \longleftrightarrow \frac{d^n X(s)}{ds^n}$$

$$t^n x(t) \longleftrightarrow (-1)^n \frac{d^n X(s)}{ds^n}$$

Ex Find $R(s)$ if, $r(t) = t u(t)$

$$u(t) \leftrightarrow \frac{1}{s}$$

$$t u(t) \leftrightarrow (-1)^1 \frac{d}{ds} \left(\frac{1}{s} \right)$$

$$\frac{d}{ds} s^{-1} = -1 \cdot \frac{1}{s^2}$$

$$R(s) = \frac{1}{s^2} \quad (\text{ramp in } s\text{-domain})$$

Ex 2 Find $X(s) \Rightarrow x(t) = t^n u(t)$

rule $\Rightarrow$ $t^n u(t) \leftrightarrow \frac{n!}{s^{n+1}}$ $u(t)$ already in s

if $x(t) = t^3 u(t)$

$$\hookrightarrow X(s) = \frac{3!}{s^4} = \frac{3 \times 2 \times 1}{s^4} = \frac{6}{s^4} \quad \#$$

Ex 3 $x(t) = t^2 e^{-at}$ find $X(s)$

$$e^{-at} \leftrightarrow \frac{1}{s+a}$$

$$t^2 e^{-at} \leftrightarrow (-1)^2 \frac{d^2}{ds^2} \left[\frac{1}{s+a} \right]$$

$$\begin{aligned} & ((s+a)^{-1})' \\ &= -(s+a)^{-2} \cdot 1 \\ &+ 2(s+a)^{-3} \cdot 1 \end{aligned}$$

$$X(s) = \frac{2}{(s+a)^3} \quad \#$$

[6] Convolution

[10]

If

$$x(t) \longleftrightarrow X(s)$$

$$h(t) \longleftrightarrow H(s)$$

then

$$x(t) * h(t) \longleftrightarrow X(s) \cdot H(s)$$

Ex 2 $x(t) = e^{-2t} u(t)$, $y(t) = \frac{2}{3}(e^{-t} + e^{-2t} + (-e^{-3t})) u(t)$

Find $h(t)$

$$y(t) = x(t) * h(t)$$

$$Y(s) = X(s) \cdot H(s)$$

$$H(s) = \frac{Y(s)}{X(s)}$$

$$\Rightarrow X(s) = \frac{1}{s+2}$$

$$Y(s) = \frac{2}{3} \left[\frac{1}{s+1} + \frac{1}{s+2} + \frac{-1}{s+3} \right]$$

للجواب
بالطريقة

$$Y(s) = \frac{2}{3} \left[\frac{s+2}{(s+1)(s+2)} + \frac{-1}{s+3} \right]$$

$$= \frac{2}{3} \left[\frac{((s+2) + (s+1))(s+3) - (s+1)(s+2)}{(s+1)(s+2)(s+3)} \right]$$

$$Y(s) = \frac{2}{3} \left[\frac{s^2 + 5s + 6 + (s^2 + 4s + 3) - (s^2 + 3s + 2)}{(s+1)(s+2)(s+3)} \right]$$

$$Y(s) = \frac{2}{3} \left[\frac{s^2 + 6s + 7}{(s+1)(s+2)(s+3)} \right]$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{2}{3} \left[\frac{s^2 + 6s + 7}{(s+1)(s+3)} \right]$$

III

↳ Now, By using inverse Laplace transform we can get $h(t)$.

[7] Initial - Value Theorem

$$x(0) = \lim_{s \rightarrow \infty} s X(s)$$

$$x(0) = \lim_{t \rightarrow 0} x(t)$$

Ex:

$$X(s) = \frac{-3s^2 + 2}{s^3 + s^2 + 3s + 2} \quad \text{Find } x(0)$$

$$x(0) = \lim_{s \rightarrow \infty} s \cdot X(s) = \lim_{s \rightarrow \infty} \frac{-3s^3 + 2s}{s^3 + s^2 + 3s + 2}$$

قسم كل شيء إلى 8

$$= \lim_{s \rightarrow \infty} \frac{-3 + 2/s^2}{1 + 1/s + 3/s^2 + 2/s^3} = \frac{-3}{1} = \boxed{-3}$$

هذه الطريقة تستخدم لإيجاد القيمة الأولية $x(0)$ في المجال الزمني (time domain) دون الحاجة إلى تحويل الدالة.

[8] Final value Theorem

[12]

$$X(\infty) = \lim_{t \rightarrow \infty} X(t) = \lim_{s \rightarrow 0} s X(s)$$

- Final value Theorem exists only if the system is stable
- it is very useful ~~theorem~~, by use it we can find the final value of the output without solving for time domain

Final value ~~of~~ system ~~is~~ zero ~~if~~ poles

Ex1

$$X(s) = \frac{5}{s(s^2 + s + 2)} \quad \text{find } X(\infty)$$

$$X(\infty) = \lim_{s \rightarrow 0} s X(s) = \lim_{s \rightarrow 0} \frac{5}{s^2 + s + 2} = \boxed{\frac{5}{2}}$$

Ex2

$$X(s) = \frac{3}{s(s-2)} \quad \text{find } X(\infty)$$

$$X(\infty) = \lim_{s \rightarrow 0} s X(s) = \lim_{s \rightarrow 0} \frac{3}{s-2} = -\frac{3}{2}$$

unstable $\leftarrow$ system ~~is~~ not stable

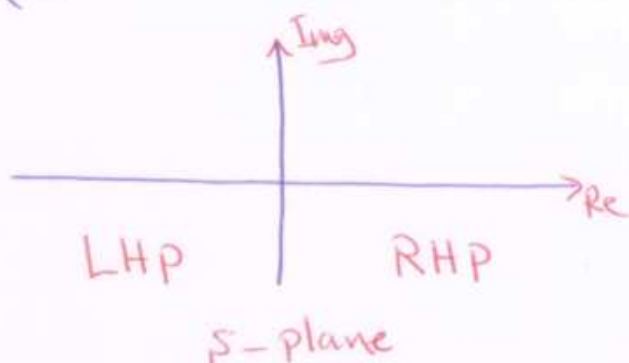
- How we can test the stability of any system?!

④ Stability

- In time domain we saw how we can determine if the system is stable or not.
- how we can do in s -domain $H(s)$ or T.F

[1] Stable

if All poles in LHP of s -plane



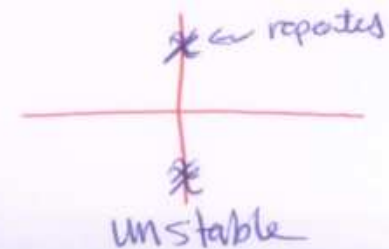
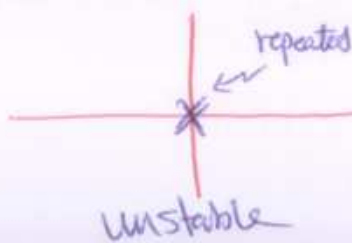
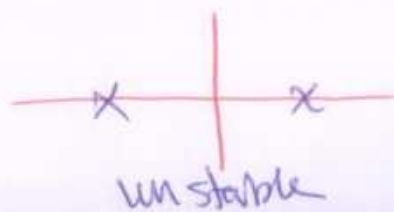
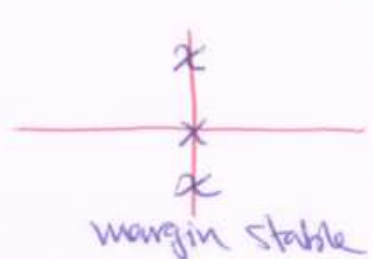
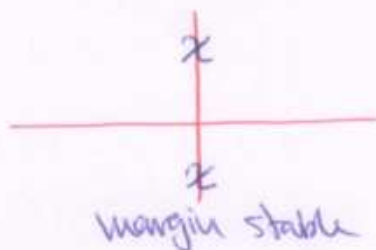
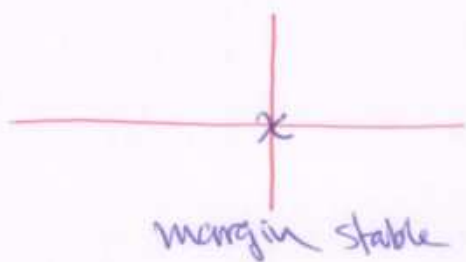
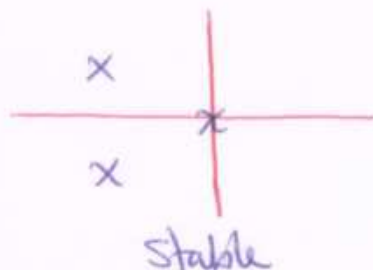
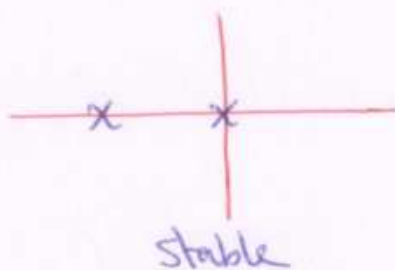
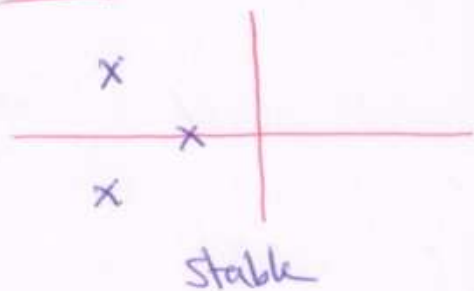
[2] margin

if at least one single pole exist on $j\omega$ axis, but no repeated poles on $j\omega$.

[3] Unstable

if at least one poles in RHP or repeated poles on $j\omega$ -axis.

Examples



⇒ how we can find the poles of the system?

$$H(s) = \frac{(s+a)(s+b) \text{ --- zeros}}{(s+c)(s+d)(s+e) \text{ --- poles}}$$

poles 1) not stable, no mp, just flat 0L

Ex 1

$$X(s) = \frac{5}{s(s^2 + s + 2)}$$

find $x(\infty)$

$$s(s^2 + s + 2) = 0$$

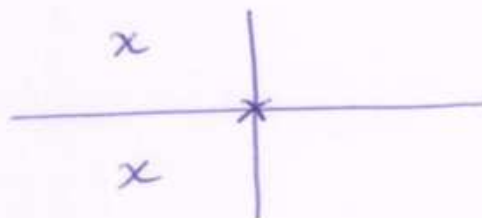
$$s_{1,2} = \frac{-1 \pm \sqrt{1^2 - 4 \times 1 \times 2}}{2 \times 1}$$

$$\therefore s(s + 0.5 - j1.325)(s + 0.5 + j1.325)$$

$$s_{1,2} = -0.5 \pm j1.325$$

$$P_1 = 0, P_2 = -0.5 + j1.325, P_3 = -0.5 - j1.325$$

stable system



$$\therefore x(\infty) = \lim_{s \rightarrow 0} s X(s)$$

$$= \lim_{s \rightarrow 0} \frac{5}{s^2 + s + 2} = \left[\frac{5}{2} \right]$$

this is the final value

Ex 2

$$X(s) = \frac{6}{s(s^2 + 25)}$$

find $x(\infty)$

$$P_1 = 0, P_2 = +j5, P_3 = -j5$$

$$s_{1,2} = \pm j\sqrt{25}$$

$$s_{1,2} = \pm j5$$

this system is marginal stable

so, there is no final value



* Inverse Laplace transform 2

- The function $X(s)$ has to be a proper rational function to find the inverse of laplace transform
 proper $\Rightarrow$ order den $>$ order num
 improper $\Rightarrow$ order num $\geq$ order den (need long division)
- The Basic procedure is to express $X(s)$ as a summation of term whose inverse laplace transform are available in a table.
- There are four general forms of solving the partial fraction, the roots of $D(s)$ are:
 - 1) Real and distinct
 - 2) Complex and distinct
 - 3) Real and Repeated
 - 4) Complex and Repeated

o long division 2

$$X(s) = \frac{2s^3 + 2s - 10}{s^3 + 3s^2 - s - 3}$$

Improper T.F

$$\begin{array}{r} 2 \\ (s^3 + 3s^2 - s - 3) \overline{) 2s^3 + 2s - 10} \\ \underline{+ 2s^3 + 6s^2 - 2s - 6} \\ -6s^2 + 2s + 6 \end{array}$$

$$\therefore X(s) = 2 + \frac{-6s^2 + 2s + 6}{s^3 + 3s^2 - s - 3}$$

From table direct $2\delta(t)$
 $\Downarrow$
P.F then use table.

(I) Real and distinct

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Ex 2

$$X(s) = \frac{10(s+1)}{s^2+4s+3}$$

find $x(t)$ & final value

By using partial fraction

$$X(s) = \frac{10(\cancel{s+1})}{(\cancel{s+1})(s+3)}$$

$$s_{1,2} = \frac{-4 \pm \sqrt{4^2 - 4 \times 1 \times 3}}{2 \times 1}$$

$$s_1 = -1$$

$$s_2 = -3$$

here we have zero-poles cancellation

$$X(s) = \frac{10}{s+3}$$

By using the table

$$x(t) = 10 e^{-3t} u(t)$$

$$x(\infty) = \lim_{t \rightarrow \infty} 10 e^{-3t} = 0$$

or

$$x(\infty) = \lim_{s \rightarrow 0} s X(s) = \lim_{s \rightarrow 0} \frac{10(s^2+s)}{s^2+4s+3} = 0$$

because there is no force input for this system it just has natural response only. and its goes to zero as t goes to infinity.

[2] Real and repeated 2

[7]

Ex2 $X(s) = \frac{2s^2 + 3s - 1}{(s+1)(s+2)^3}$

find $x(t)$

Stable
All poles in LHP

By using partial fraction.

جس میں دو الگ الگ پکڑے
کے ہیں اور جگہ جگہ پر

$$X(s) = \frac{k_1}{s+1} + \frac{k_2}{(s+2)^3} + \frac{k_3}{(s+2)^2} + \frac{k_4}{(s+2)}$$

$$k_1 = \left. \frac{2s^2 + 3s - 1}{(s+2)^3} \right|_{s=-1} \Rightarrow k_1 = \boxed{-2}$$

$$k_2 = \left. \frac{2s^2 + 3s - 1}{(s+1)} \right|_{s=-2} \Rightarrow k_2 = \boxed{-1}$$

$$k_3 = \frac{1}{1!} \cdot \left. \frac{d}{ds} [(s+2)^2 X(s)] \right|_{s=-2} = \frac{(4s+3)(s+1) - (2s^2+3s-1) \cdot 1}{(s+1)^2} \bigg|_{s=-2}$$

$$k_3 = \boxed{4}$$

$$k_4 = \frac{1}{2!} \cdot \left. \frac{d^2}{ds^2} [(s+2)^3 X(s)] \right|_{s=-2} = \frac{(4s+4)(s+1)^2 - (2s^2+4s+4)(2) \cdot (s+1)}{((s+1)^2)^2} \bigg|_{s=-2}$$

$$k_4 = \frac{1}{2!} \cdot 4 = \boxed{2}$$

$$\therefore X(s) = \frac{-2}{s+1} - \frac{1}{(s+2)^3} + \frac{4}{(s+2)^2} + \frac{2}{(s+2)}$$

$$x(t) = \left[-2e^{-t} - t^2 e^{-2t} + 4t e^{-2t} + 2e^{-2t} \right] u(t)$$

[3] Simple complex poles:-

[18]

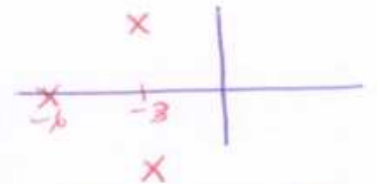
Ex:

$$X(s) = \frac{100(s+3)}{(s+6)(s^2+6s+25)}$$

$$s_{1,2} = \frac{-6 \pm \sqrt{36 - 4 \times 1 \times 25}}{2}$$

$$s_{1,2} = -3 \pm j4$$

$$X(s) = \frac{100(s+3)}{(s+6)(s+3-j4)(s+3+j4)}$$



stable system

$$X(s) = \frac{k_1}{s+6} + \frac{k_2}{s+3-j4} + \frac{k_2^*}{s+3+j4}$$

residue at pole

$$k_1 = \left. \frac{100(s+3)}{s^2+6s+25} \right|_{s=-6} \Rightarrow k_1 = -12$$

$$k_2 = \left. \frac{100(s+3)}{(s+6)(s+3+j4)} \right|_{s=-3-j4} \Rightarrow k_2 = 10 \angle -53.13$$

$$\therefore k_2^* = 10 \angle 53.13$$

$$\therefore X(s) = \frac{-12}{(s+6)} + \frac{10 \angle -53.13}{s+3-j4} + \frac{10 \angle 53.13}{s+3+j4}$$

Formula 2

$$\mathcal{L}^{-1} \left\{ \frac{k}{s+\alpha-j\beta} + \frac{k^*}{s+\alpha+j\beta} \right\} \Rightarrow \underline{k = |k| \angle \theta}$$

$$= 2|k| e^{-\alpha t} \cos(\beta t + \theta)$$

نصف الزاوية في k والزاوية في θ هي الزاوية التي يكون عليها Imag في k

$$\therefore x(t) = \left[-12 e^{-6t} + 20 e^{-3t} \cos(4t - 53.13) \right] u(t)$$

[4] Repeated complex poles

[19]

Ex2

$$X(s) = \frac{768}{(s^2 + 6s + 25)^2}$$

find $x(t)$

stable

$$X(s) = \frac{k_1}{(s+3-j4)^2} + \frac{k_2}{(s+3-j4)} + \frac{k_1^*}{(s+3+j4)^2} + \frac{k_2^*}{(s+3+j4)}$$

$$k_1 = \left. \frac{768}{(s+3+j4)^2} \right|_{s=-3+j4} = -12 \angle 0$$

$$k_2 = \frac{1}{1!} \left. \frac{d}{ds} [(s+3-j4)^2 X(s)] \right|_{s=-3+j4} = \left. \frac{0 - 768(2 \cdot (s+3+j4) \cdot 1)}{((s+3+j4)^2)^2} \right|_{s=-3+j4}$$

$$k_2 = 3 \angle -90^\circ$$

$$\therefore k_1^* = -12 \angle 0, \quad k_2^* = 3 \angle +90^\circ$$

$$\Rightarrow X(s) = \frac{-12 \angle 0}{(s+3-j4)^2} + \frac{3 \angle -90}{(s+3-j4)} - \frac{12 \angle 0}{(s+3+j4)^2} + \frac{3 \angle 90}{(s+3+j4)}$$

Formula: $\mathcal{L}^{-1} \left\{ \frac{k}{(s+\alpha-j\beta)^r} + \frac{k^*}{(s+\alpha+j\beta)^r} \right\} \quad k = |k| \angle \theta$

$$= \frac{2|k|}{(r-1)!} t^{r-1} e^{-\alpha t} \cos(\beta t + \theta)$$

$$\therefore x(t) = \left[\frac{-24}{1!} t e^{-3t} \cos(4t) + 6 e^{-3t} \cos(4t - 90^\circ) \right] u(t)$$

* Transfer function 2

20

→ it is the relationship between the input and output of the system $H(s)$

→ T.F $H(s) = \frac{Y(s)}{X(s)}$



→ it is useful in determining the stability of a system

→ to find it put all initial values are zero

في حال معرفة $H(s)$ نحدد stability من خلال poles في s إذا كانت الأجزاء الحقيقية سالبة فإن النظام مستقر، وإذا كانت موجبة فإن النظام غير مستقر.

Ex2 $2y''(t) + 3y'(t) + 5y(t) = x'(t) - 2x(t)$

Find the T.F? all initial value zero

$H(s)$?! 1st convert D.eq to Frequency domain

$$2s^2 Y(s) + 3s Y(s) + 5 Y(s) = s X(s) - 2 X(s)$$

$$\Rightarrow Y(s) (2s^2 + 3s + 5) = X(s) (s - 2)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{s - 2}{2s^2 + 3s + 5}$$

ℒ

⊗ Solution of Differential Equations 2

21

By using Differential property in table

Ex 2 $y''(t) + 4y'(t) + 3y(t) = 2x''(t) - 4x'(t) - x(t)$

$y(0) = -2, y'(0) = 1, x(t) = u(t)$ find $y(t)$?

$$[s^2 y(s) - s y(0) - \overset{1}{\cancel{y'(0)}}] + 4[s y(s) - \overset{-2}{\cancel{y(0)}}] + 3y(s) = 2[s^2 x(s)] - 4s x(s) - x(s)$$

$$Y(s) [s^2 + 4s + 3] + \underbrace{2s + 7}_{\text{initial condition term}} = X(s) (\underbrace{2s^2 - 4s - 1}_{\text{input term}})$$

$\Rightarrow x(t) = u(t) \Rightarrow X(s) = \frac{1}{s}$

$$Y(s) (s^2 + 4s + 3) = \frac{1}{s} (2s^2 - 4s - 1) - 2s - 7$$

$$Y(s) (s^2 + 4s + 3) = \cancel{2s} - 4 - \frac{1}{s} - \cancel{2s} - 7 = -11 - \frac{1}{s}$$

$$Y(s) = \frac{\underbrace{-11 - \frac{1}{s}}_{\text{polynomial in } s \text{ in numerator}}}{s^2 + 4s + 3}$$

$$Y(s) = \frac{-11s - 1}{s(s^2 + 4s + 3)} = \frac{-11s - 1}{s(s+3)(s+1)}$$

$$= \frac{k_1}{s} + \frac{k_2}{s+3} + \frac{k_3}{s+1}$$

$k_1 = -\frac{1}{3}, k_2 = \frac{16}{3}, k_3 = -5$

$$Y(s) = \frac{-\frac{1}{3}}{s} + \frac{\frac{16}{3}}{s+3} + \frac{-5}{s+1}$$

$$y(t) = \left[-\frac{1}{3} + \frac{16}{3} e^{-3t} - 5 e^{-t} \right] u(t)$$

⊗ Analysis of Electrical networks

22

Notes

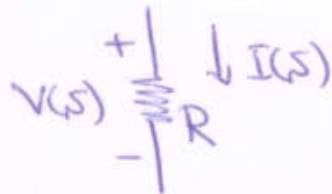
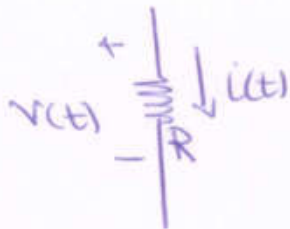
The Laplace transform method gives the total response which includes zero-input and zero-state components.

It is possible to separate the two components if we so desire. (See page 37, 38)

Time Domain

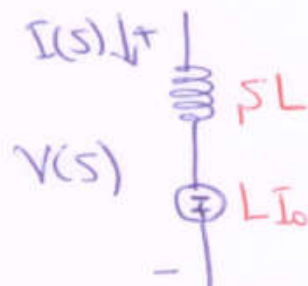
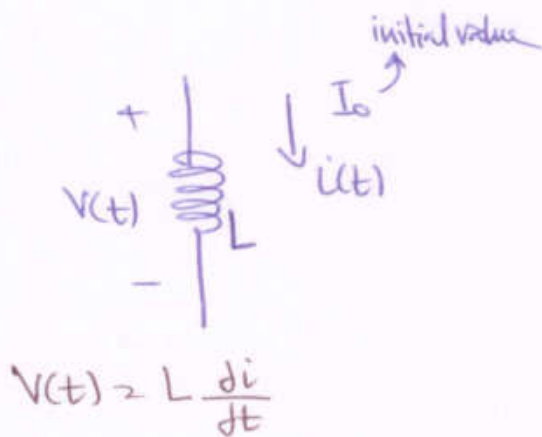
s-Domain

1



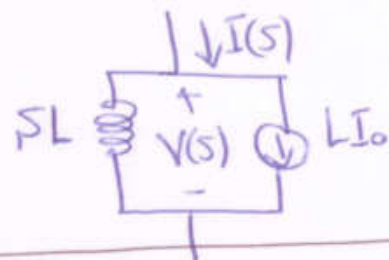
$$V(s) = R I(s)$$

2



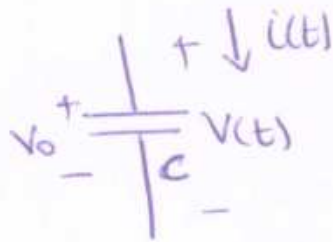
$$V(s) = L [s I(s) - I_0]$$

OR



$$I(s) = \frac{V(s)}{sL} + \frac{I_0}{s}$$

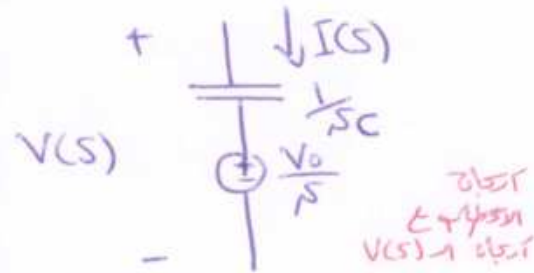
[3]



$$i = C \frac{dV}{dt}$$

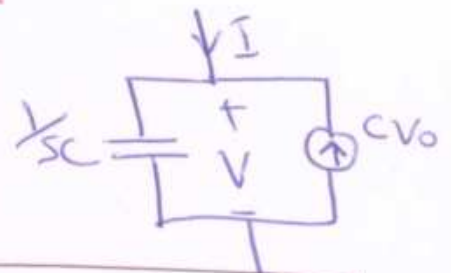
$$V = \frac{1}{C} \int_0^t i(t) dt + V_0$$

[23]



$$V(s) = \frac{I(s)}{sC} + \frac{V_0}{s}$$

OR



$$I(s) = sC V - C V_0$$

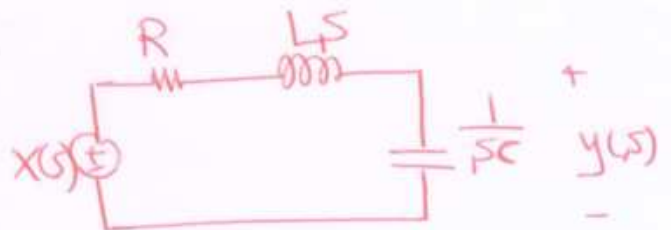
Ex2

$$H(s) = \frac{Y(s)}{X(s)}$$

$$= \frac{I(s) \cdot \frac{1}{sC}}{I(s) \left(R + Ls + \frac{1}{sC} \right)} \cdot \frac{sC}{sC}$$

$$= \frac{1}{R s C + L s^2 C + 1}$$

$$H(s) = \frac{\frac{1}{LC}}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$



LC circuit
is a stable system

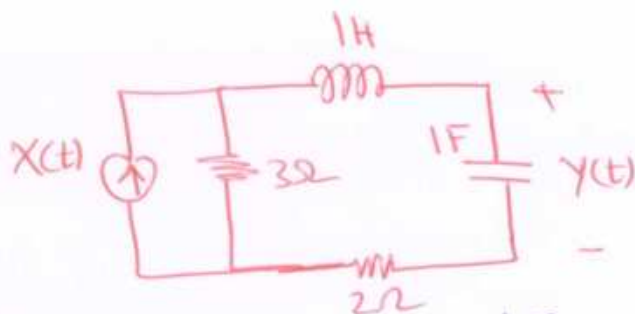
stable system

Ex 1

$$V_c(0) = 1V$$

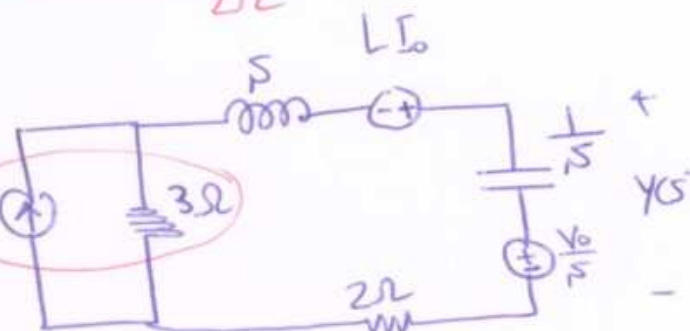
$$i_L(0) = 2A$$

$$X(t) = \delta(t) \text{ , find } y(t)?$$



دلیلش از اینست که در این مدار

$$X(s) = 1$$



by using KVL 2

$$3 - 3I(s) - 5I(s) + 2 - \frac{1}{s}I(s)$$

$$- \frac{1}{s} - 2I(s) = 0$$

$$I(s)(3 + 5 + \frac{1}{s} + 2) = 3 + 2 - \frac{1}{s}$$

$$I(s) \left(5 + \frac{s^2 + 1}{s} \right) = \frac{5s - 1}{s}$$

$$I(s)(5s + s^2 + 1) = 5s - 1$$

$$I(s) = \frac{5s - 1}{(s^2 + 5s + 1)}$$

$$\therefore Y(s) = \frac{1}{s} I(s) + \frac{1}{s}$$

$$Y(s) = \frac{5s - 1}{s(s^2 + 5s + 1)} + \frac{1}{s}$$

$$Y(s) = \frac{s + 10}{s^2 + 5s + 1}$$

Now, By using partial fraction you can find $y(t)$

$$Y(s) = \frac{k_1}{(s+0.21)} + \frac{k_2}{(s+4.79)}$$

$$s_{1,2} = \frac{-5 \pm \sqrt{25 - 4 \times 1 \times 1}}{2 \times 1}$$

$$k_1 = \left. \frac{s+10}{s+4.79} \right|_{s=-0.21} = 2.138$$

$$s_{1,2} = -2.5 \pm 2.29$$

$$s_{1,2} = 0.21 \text{ \& } 4.79$$

$$k_2 = \left. \frac{s+10}{(s+0.21)} \right|_{s=-4.79} = -1.138$$

$$Y(s) = \frac{2.138}{(s+0.21)} - \frac{1.138}{(s+4.79)}$$

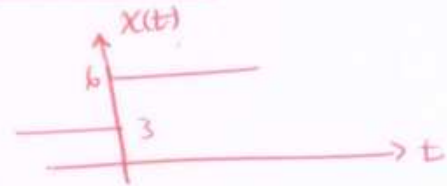
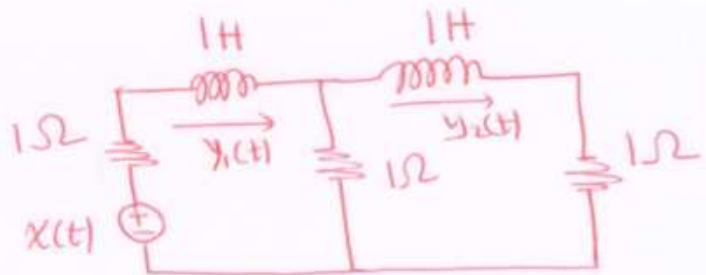
$$y(t) = \left[2.138 e^{-0.21t} - 1.138 e^{-4.79t} \right] u(t) \quad \#$$

Ex2

Find $y_1(t)$, $y_2(t)$ for $t \geq 0$

$$y_1(0) = 2, \quad y_2(0) = 1$$

$$\text{For } t \geq 0 \quad x(t) = 6 u(t)$$

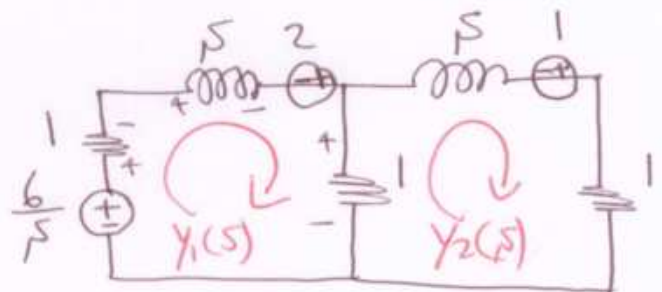


By using mesh current

$$+Y_1(s) + sY(s) + (Y_1(s) - Y_2(s)) \cdot 1 = \frac{6}{s} + 2$$

$$Y_1(s)(s+2) - Y_2(s) = \frac{6}{s} + 2$$

$$Y_2(s)(s+2) - Y_1(s) = 1$$



$$Y_1(s)(s+2) - Y_2(s) = \frac{6+2s}{s}$$

$$-Y_1(s) + Y_2(s)(s+2) = 1$$

$$\begin{bmatrix} s+2 & -1 \\ -1 & s+2 \end{bmatrix} \begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \begin{bmatrix} \frac{6+2s}{s} \\ 1 \end{bmatrix}$$

رَبِّ الْمُنْتَفِلِينَ
وَأَنزَلَ مِنْ سَمَاءٍ مَاءً

Cramer's rule

$$AX = b$$

↓
مصفوفة المصفوفة
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مصفوفة المصفوفة

$$X_i = \frac{\det(A_i)}{\det(A)}$$

where $i = 1, 2, \dots, n$

A_i is the matrix formed by replacing the i th column of A by the column vector B

$$\det(A) = \begin{vmatrix} s+2 & -1 \\ -1 & s+2 \end{vmatrix} = (s+2)(s+2) - (-1 \times -1) = s^2 + 4s + 3$$

$$Y_1(s) = \frac{\begin{vmatrix} \frac{2s+6}{s} & -1 \\ +1 & s+2 \end{vmatrix}}{\det(A)} = \frac{(\frac{2s+6}{s})(s+2) + 1}{s^2 + 4s + 3} = \frac{\frac{2s^2 + 10s + 12 + s}{s}}{s^2 + 4s + 3}$$

$$Y_2(s) = \frac{\begin{vmatrix} s+2 & \frac{2s+6}{s} \\ -1 & 1 \end{vmatrix}}{\det(A)} = \frac{(s+2) + (\frac{2s+6}{s})}{s^2 + 4s + 3} = \frac{\frac{s^2 + 2s + 2s + 6}{s}}{s^2 + 4s + 3} = \frac{s^2 + 4s + 6}{s(s^2 + 4s + 3)}$$

∴ By find inverse Laplace transform $Y_1(s)$ & $Y_2(s)$

$$y_1(t) = \left(4 - \frac{3}{2}e^{-t} - \frac{1}{2}e^{-3t} \right) u(t)$$

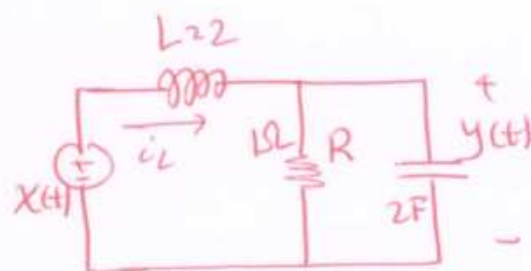
$$y_2(t) = \left(2 - \frac{3}{2}e^{-t} + \frac{1}{2}e^{-3t} \right) u(t)$$

✓

⊗ Ex 2

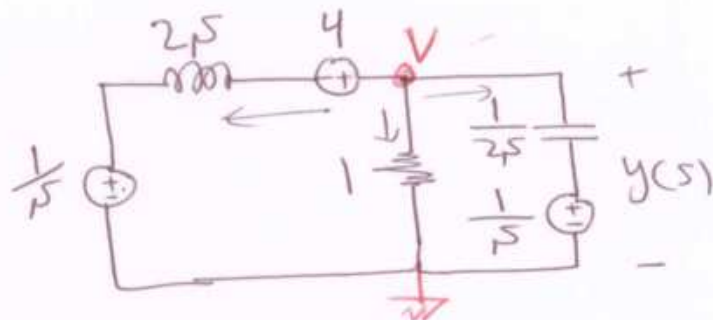
$V_C(0) = 1$, $i_L(0) = 2$, $x(t) = u(t)$

find $y(t)$?



another way:

node voltage



$$\frac{V - 4 - \frac{1}{s}}{2s} + \frac{V}{1} + \frac{V - \frac{1}{s}}{\frac{1}{2s}} = 0$$

$$V = y(s)$$

$$\frac{y(s)}{2s} - \frac{4}{2s} - \frac{1}{2s^2} + y(s) + 2s y(s) - 2 = 0$$

$$y(s) \left[\frac{1}{2s} + 1 + 2s \right] = \frac{1}{2s^2} + \frac{4}{2s} + 2 \quad \text{input given } \frac{1}{2s^2} \text{ s}$$

$$y(s) [s + 2s^2 + 4s^3] = 1 + 4s + 4s^2$$

$$y(s) = \frac{4s^2 + 4s + 1}{4s^3 + 2s^2 + s} = \frac{4s^2 + 4s + 1}{s(4s^2 + 2s + 1)}$$

$$y(s) = \frac{K_1}{s} + \frac{K_2}{s + 0.25 - j0.433} + \frac{K_2^*}{s + 0.25 + j0.433} \left\{ \begin{array}{l} s_{1,2} = \frac{-2 \pm \sqrt{4 - 4 \times 4}}{2 \times 4} \\ s_{1,2} = -0.25 \pm j0.433 \end{array} \right.$$

$$K_1 = \left. \frac{4s^2 + 4s + 1}{4s^2 + 2s + 1} \right|_{s=0} = \boxed{1}$$

$$k_2 = \frac{4s^2 + 4s + 1}{s(s + 0.25 + j0.433)} \bigg|_{s = -0.25 + j0.433}$$

$$k_2 = 2.31 \angle -90$$

$$k_2^* = 2.31 \angle 90$$

$$\therefore Y(s) = \frac{1}{s} + \frac{2.31 \angle -90}{s + 0.25 - j0.433} + \frac{2.31 \angle 90}{s + 0.25 + j0.433}$$

$$y(t) = u(t) + 4.62 e^{-0.25t} \cos(0.433t - 90) u(t)$$

⌘

* Additional exercises

1) Compute the LT of the following function:

a) $x(t) = 4 \sin(100t) u(t)$

By using table $\sin \omega t \leftrightarrow \frac{\omega}{s^2 + \omega^2}$

$$X(s) = \frac{400}{s^2 + (100)^2}$$

b) $x(t) = 4 \sin(100t - 10) u(t - 0.1)$

$$x(t) = 4 \sin(100(t - 0.1)) u(t - 0.1)$$

By using table & properties

$$4 \sin(100(t)) \leftrightarrow \frac{400}{s^2 + 100^2}$$

shift in t.D $4 \sin(100(t - 0.1)) \leftrightarrow \frac{400}{s^2 + 100^2} \cdot e^{-0.1s}$

$$\therefore X(s) = \frac{400}{s^2 + 100} \cdot e^{-0.1s} \quad \#.$$

c) $x(t) = u(t) - e^{-2t} \cos(10t) u(t)$

$$u(t) \leftrightarrow \frac{1}{s}$$

$$\cos 10t \leftrightarrow \frac{s}{s^2 + 10^2}$$

$$e^{-2t} \cos 10t \leftrightarrow \frac{(s+2)}{(s+2)^2 + 10^2}$$

2) Find the inverse Laplace transform for the following 2

a) $X(s) = \frac{10(s+1)}{s^2 + 4s + 3} \cdot e^{-2s}$

By using partial fraction $\hookrightarrow$

$$X'(s) = \frac{10(s+1)}{(s+3)(s+1)} = \frac{10}{s+3}$$

$$x'(t) = 10 e^{-3t} u(t)$$

$$X(s) = \frac{10}{s+3} \cdot \left(e^{-2s} \right) \rightarrow$$

shift in time domain

$$x(t) = 10 e^{-3(t-2)} u(t-2)$$

#.

$$b) \quad X(s) = \frac{10(s+1)}{(s^2+4s+8)s}$$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$X(s) = \frac{10(s+1)}{s((s+2)^2+4)}$$

$$s_{1,2} = \frac{-4 \pm \sqrt{16 - 4 \times 8 \times 1}}{2 \times 1}$$

$$s_{1,2} = -2 \pm j2$$

$$X(s) = \frac{10(s+1)}{s(s+2-j2)(s+2+j2)}$$

$$X(s) = \frac{k_1}{s} + \frac{k_2}{s+2-j2} + \frac{k_2^*}{s+2+j2}$$

$$k_1 = \left. \frac{10(s+1)}{s^2+4s+8} \right|_{s=0} = \boxed{1.25}$$

$$k_2 = \left. \frac{10(s+1)}{s(s+2+j2)} \right|_{s=-2-j2} = \underline{1.976 \angle -108.43}$$

$$k_2^* = 1.976 \angle 108.43$$

$$\therefore X(s) = \frac{1.25}{s} + \frac{1.976 \angle -108.43}{s+2-j2} + \frac{1.976 \angle 108.43}{s+2+j2}$$

$$x(t) = \left(1.25 + 3.95 e^{-2t} \cos(2t - 108.43) \right) u(t)$$

✓

3) Find the final value of the following function. 31

a) $X(s) = \frac{10(s+1)}{s(s^2+4s+8)}$

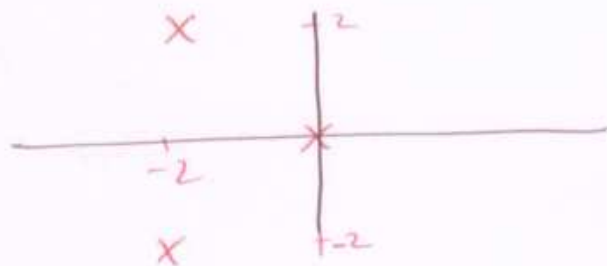
find the poles of characteristic equation as

$$s(s^2+4s+8) = 0$$

$$s_1 = 0 \quad \& \quad s_2 = -2 + j2 \quad \& \quad s_3 = -2 - j2$$

Now mapping these poles on s-plane to determine the stability -

stable system



So

$$X(\infty) = \lim_{s \rightarrow 0} sX(s)$$

$$= \lim_{s \rightarrow 0} \frac{10(s+1)}{s^2+4s+8} = \frac{10}{8} = \boxed{1.25}$$

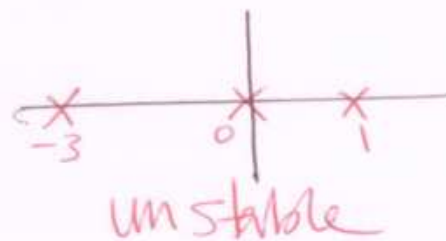
$\mathcal{H}$

b) $X(s) = \frac{10(s+1)}{(s^2+2s-3)s}$

$$= \frac{-2 \pm \sqrt{4 - 4 \times 1 \times -3}}{2}$$

$$s(s^2+2s-3) = 0 \Rightarrow s_1 = 0 \quad \& \quad s_2 = -3 \quad \& \quad s_3 = +1$$

So there is no final value because this signal doesn't stable system $\mathcal{H}$.



4) Solve the following D. eq using Laplace transform. 32

a) $y'(t) + 4y(t) = 3x(t)$

$x(t) = \sin(2t)$, $y(0) = 0$

$$(sY(s) - Y(0)) + 4Y(s) = 3 \cdot \frac{2}{s^2 + 4}$$

$$Y(s)(s+4) = \frac{6}{s^2+4} + 1$$

$$Y(s) = \frac{6 + s^2 + 4}{(s^2+4)(s+4)}$$

$$Y(s) = \frac{s^2 + 10}{(s+4)(s+j2)(s-j2)}$$

$$Y(s) = \frac{k_1}{(s+4)} + \frac{k_2}{(s-j2)} + \frac{k_2^*}{(s+j2)}$$

$$k_1 = \left. \frac{s^2+10}{(s^2+4)} \right|_{s=-4} = 0.5$$

$$k_2 = \left. \frac{s^2+10}{(s+4)(s+2+j2)} \right|_{s=-2-j2} = \frac{1.132}{\angle -173.66}$$

$$y(t) = \left(0.5 e^{-4t} + 2.26 e^{-0t} \cos(2t - 173.66) \right) u(t)$$

$$c) \quad y''(t) + 7y'(t) + 12y(t) = 6x \quad ; \quad x = u(t)$$

$$y(0) = 0, \quad y'(0) = -2$$

33

$$\left[s^2 Y(s) - \cancel{s y(0)} - \overset{-2}{y'(0)} \right] + 7 \left[s Y(s) - \cancel{y(0)} \right] + 12 Y(s) = 6 \frac{1}{s}$$

$$s^2 Y(s) + 2 + 7s Y(s) + 12 Y(s) = \frac{6}{s}$$

$$Y(s) (s^2 + 7s + 12) = \frac{6}{s} - 2 = \frac{6 - 2s}{s}$$

$$Y(s) = \frac{6 - 2s}{s(s^2 + 7s + 12)}$$

$$\frac{-7 \pm \sqrt{7^2 - 4 \times 1 \times 12}}{2}$$

$$s_{1,2} = -3, -4$$

$$= \frac{k_1}{s} + \frac{k_2}{(s+3)} + \frac{k_3}{(s+4)}$$

$$k_1 = \left. \frac{6 - 2s}{s^2 + 7s + 12} \right|_{s=0} = 0.5$$

$$k_2 = \left. \frac{6 - 2s}{s(s+4)} \right|_{s=-3} = -\frac{4}{1}$$

$$k_3 = \left. \frac{6 - 2s}{s(s+3)} \right|_{s=-4} = +3.5$$

$$X(s) = \frac{0.5}{s} + \frac{-\frac{4}{1}}{(s+3)} + \frac{+3.5}{s+4}$$

$$x(t) = \left[\frac{1}{2} - \frac{4}{1} e^{-3t} + 3.5 e^{-4t} \right] u(t)$$

R.

4) Find the transfer function for

$$y'' + 4y' + 20y = 2x' - x$$

All initial values equal zero

$$s^2 Y(s) + 4s Y(s) + 20 Y(s) = 2s X(s) - X(s)$$

$$Y(s) (s^2 + 4s + 20) = (2s - 1) X(s)$$

$$\frac{Y(s)}{X(s)} = H(s) = \frac{2s - 1}{s^2 + 4s + 20}$$

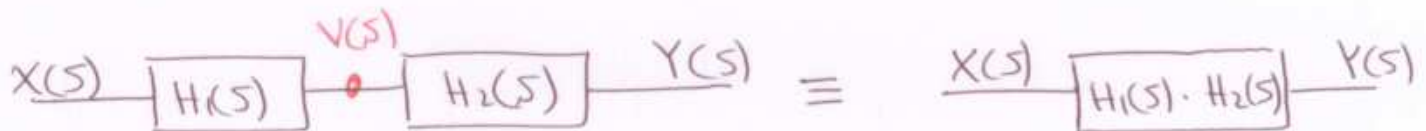
A.

* Block Diagrams

→ it is convenient to represent a system by suitably interconnected subsystems.

→ There are 3 types of interconnected subsystems.

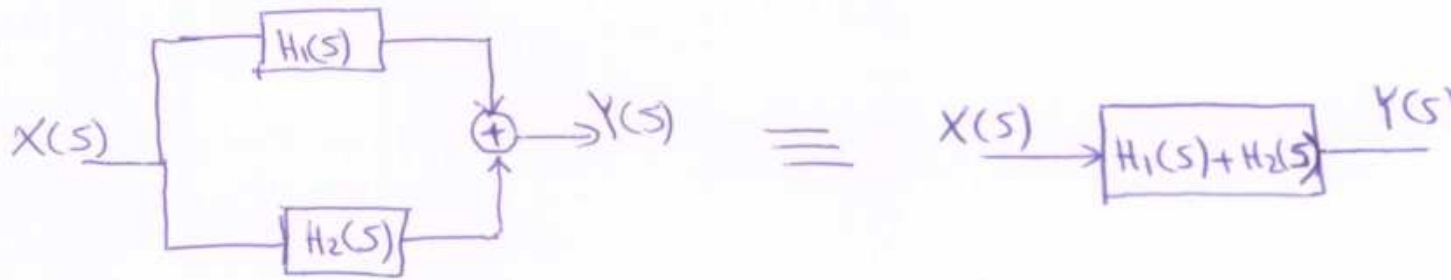
I) Cascade (series)



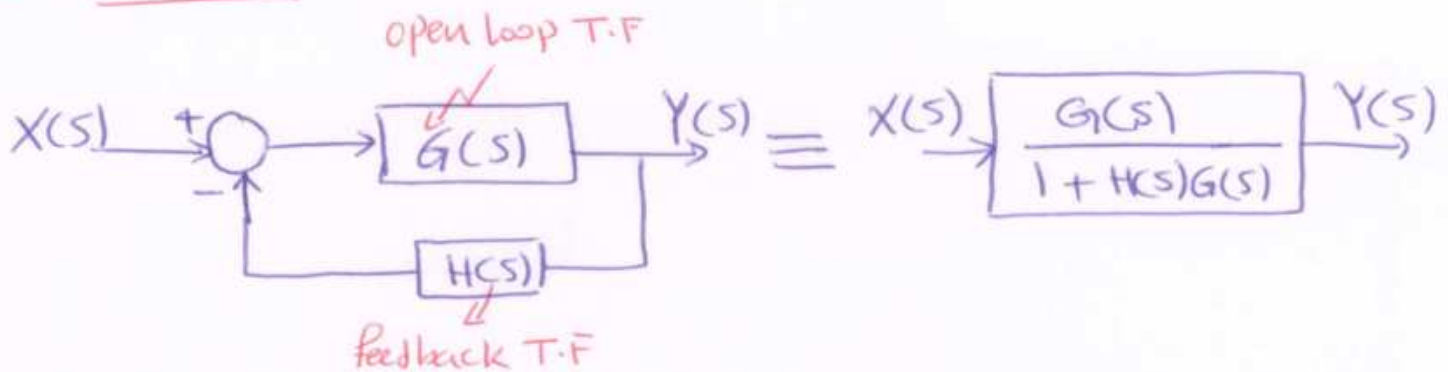
$$\frac{Y(s)}{X(s)} = \frac{V(s)}{X(s)} \overset{\text{End}}{\underset{\text{Cale}}{*}} \frac{Y(s)}{V(s)} = H_1(s) \times H_2(s)$$

[2] parallel 2

[35]



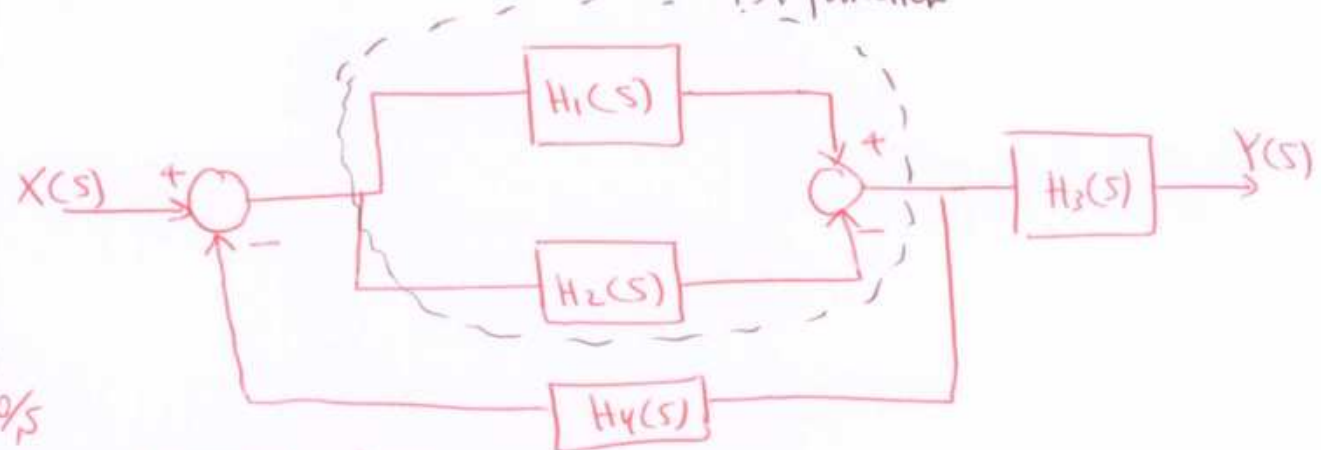
[3] feedback 2



Example 2

Ex2 Simplify the block diagram to find the T.F

1st Parallel



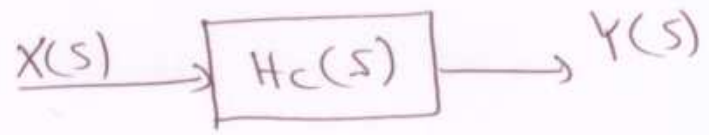
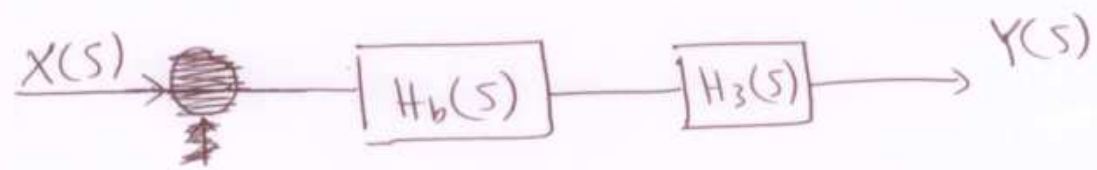
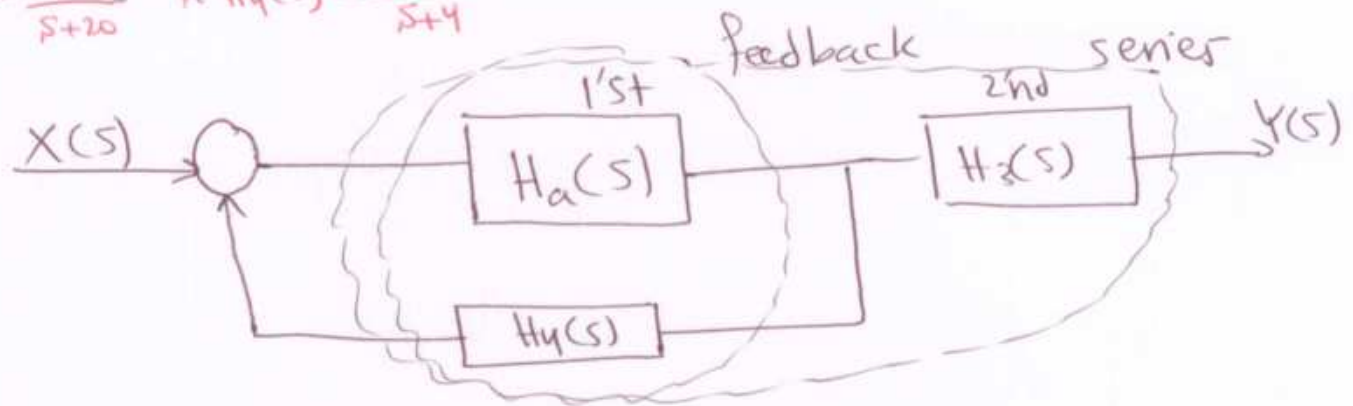
Given 2

$$H_1(s) = 2$$

$$H_2(s) = \frac{10}{s}$$

$$H_3(s) = \frac{0.1}{s+20}$$

$$\text{R } H_4(s) = \frac{2}{s+4}$$



where

$$H_a(s) = H_1(s) + H_2(s) = 2 + \frac{10}{s} = \frac{2s+10}{s}$$

$$H_b(s) = \left(\frac{H_1(s) + H_2(s)}{1 + (H_1(s) + H_2(s)) \cdot H_4(s)} \right) = \frac{\frac{2s+10}{s}}{1 + \left(\frac{2s+10}{s} \right) \cdot \frac{0.1}{s+20}}$$

$$H_c(s) = \left(\frac{H_1(s) + H_2(s)}{1 + (H_1(s) + H_2(s)) H_4(s)} \right) \cdot H_3(s)$$

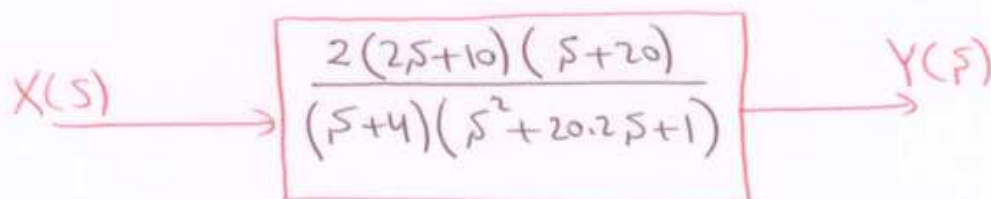
$$H_b(s) = \frac{\frac{2s+10}{s}}{1 + \frac{0.2s+1}{s(s+20)}} = \frac{(2s+10)\cancel{s}}{\cancel{s}^2 + 20s + 0.2s + 1}$$

$$H_b(s) = \frac{(2s+10)(s+20)}{s^2 + 20.2s + 1}$$

$$H_c(s) = H_b(s) \cdot H_3(s) = \frac{(2s+10)(s+20)}{s^2 + 20.2s + 1} \cdot \frac{2}{s+4}$$

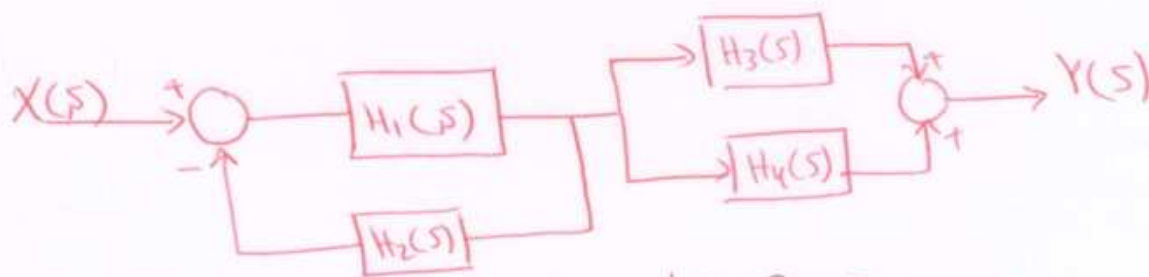
$$H(s) = H_c(s) = \frac{2(2s+10)(s+20)}{(s+4)(s^2 + 20.2s + 1)} \quad \#$$

↳ This called total transfer function of the Block Diagram



#

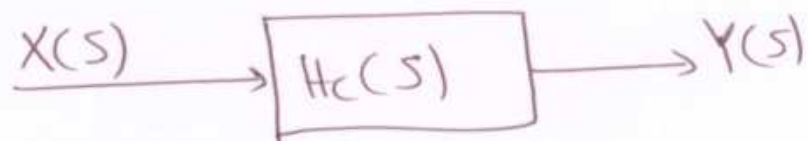
Ex 2: Reduce the block diagram to one block



① feedback

② parallel

③ cascade



where

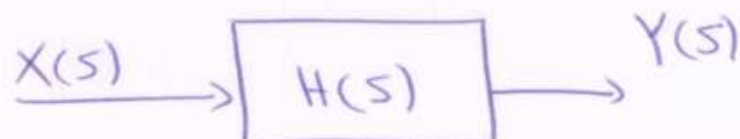
$$H_a(s) = \frac{H_1(s)}{1 + H_1(s)H_2(s)}$$

$$H_b(s) = H_3(s) + H_4(s)$$

$$H_c(s) = H_a(s) + H_b(s)$$

$$\therefore H_c(s) = \frac{H_1(s)H_3(s)}{1 + H_1(s)H_2(s)} + \frac{H_1(s)H_4(s)}{1 + H_1(s)H_2(s)}$$

$$H(s) = \frac{H_1(s)H_3(s)(1 + H_1(s)H_2(s)) + H_1(s)H_4(s)(1 + H_1(s)H_2(s))}{(1 + H_1(s)H_2(s))(1 + H_1(s)H_2(s))}$$



HP

End of chapter 4