

chapter 5Z - transform

→ Z-transform does for Discrete time systems, where the Laplace transform does for continuous time systems.

→ Z-transform is used to:

1) Solve the difference equations with initial conditions

2) Solve zero state systems using the transfer function
(means there is no initial condition)

→ As we said before there are two general form for Difference equation

a) Advanced:

$$y[n+N] + y[n+N-1] + \dots + y[n] = x[n+M] + \dots + x[n]$$

↳ it has an order equal to $\boxed{\text{Max}[N, M]}$

b) Delay form:

$$y[n] + y[n-1] + \dots + y[n-N] = x[n] + \dots + x[n-M]$$

↳ it has an order equal to $\boxed{\text{Max}[N, M]}$

* Direct Z-transform

- we define $X(z)$, the direct Z-transform of $x[n]$ as

$$X[z] = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

where,

z is the complex variable.

- we can find the inverse of Z-transform by

$$x[n] = \frac{1}{2\pi j} \oint X[z] z^{n-1} dz$$

$\oint \Rightarrow$ integration around closed path.

- The unilateral Z-transform

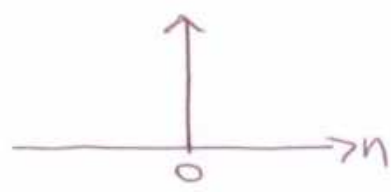
for causal signals which started from zero to positive.

$$X[z] = \sum_{n=0}^{\infty} x[n] z^{-n}$$

⊗ Z-transform of Elementary Functions.

1] Unit - sample sequence $\delta[n]$

$$\delta[n] = \begin{cases} 1 & , n=0 \\ 0 & , n \neq 0 \end{cases}$$



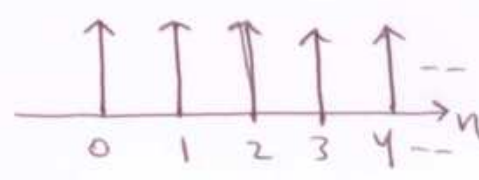
$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = 1 z^0 + 0 z^{-1} + \dots = \boxed{1}$$

$$\boxed{\delta[n] \leftrightarrow 1} \text{ rule 1}$$

2] Unit - step sequence

n : integer number

$$u[n] = \begin{cases} 1 & , n \geq 0 \\ 0 & , n < 0 \end{cases}$$



$$\begin{aligned} X(z) &= \sum_{n=0}^{\infty} x[n] z^{-n} \\ &= 1 + z^{-1} + z^{-2} + \dots \end{aligned}$$

Theorem 2

$$\boxed{1 + r + r^2 + r^3 + \dots = \sum_{n=0}^{\infty} r^n = \frac{1}{1-r} \quad |r| < 1}$$

then

$$\boxed{X(z) = \frac{1}{1-z^{-1}} = \frac{z}{z-1}}$$

$$\begin{aligned} |z^{-1}| &< 1 \\ \boxed{|z| > 1} \\ \text{ROC} \end{aligned}$$

$$\text{rule 2} \quad (u[n] \leftrightarrow \frac{z}{z-1}, |z| > 1)$$

[3]

$$x[n] = a^n u[n]$$

 called polynomial function (a^n)

[4]

$$X[z] = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} \underbrace{(a z^{-1})^n}_r$$

$$X[z] = \frac{1}{1 - a z^{-1}} \quad |a z^{-1}| < 1$$

$$|a| < |z|$$

$$\therefore \boxed{X[z] = \frac{z}{z - a}, \quad |z| > |a|}$$

[4] Exponential function 2

$$x(t) = \begin{cases} e^{-at}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

$$x[n] = e^{-anT} \quad n = 0, 1, 2, \dots$$

$\mu \rightarrow n$
 $t \rightarrow nT$
 Discrete time
 $x[n] \equiv x(nT)$

$$X[z] = \sum_{n=0}^{\infty} e^{-anT} z^{-n} = \sum_{n=0}^{\infty} \underbrace{(e^{-aT} z^{-1})^n}_r$$

$$\boxed{X[z] = \frac{z}{z - e^{-aT}} \quad |z| > e^{-aT}}$$

ROC $\frac{-aT}{asT}$

$$x[n] = \begin{cases} e^{-anT}, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

L

The z-transform just used with Discrete time only.

Ex 2

$$x[n] = \begin{cases} \left(\frac{1}{2}\right)^n & , n \geq 0 \\ 0 & , n < 0 \end{cases}$$

$$y[n] = \begin{cases} -\left(\frac{1}{2}\right)^n & , \underline{n \leq 0} \text{ not in region} \\ 0 & , n > 0 \end{cases}$$

- not in region

find $X(z) + Y(z)$

$$- X(z) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^{-n} = \sum_{n=0}^{\infty} \left(\frac{z^{-1}}{2}\right)^n$$

$$X(z) = \frac{z}{z - 1/2} \quad |z| > 1/2$$

(note)

$$- Y(z) = - \sum_{n=-\infty}^{-1} \left(\frac{1}{2}\right)^n z^{-n}$$

$$= - \sum_{m=+\infty}^{-1} \left(\frac{1}{2}\right)^{-m} z^m$$

let $n = -m$

$$= - \sum_{m=1}^{\infty} \left(\left(\frac{1}{2}\right)^{-1} z\right)^m = - \sum_{m=1}^{\infty} (2z)^m$$

من أجل أن تكون السلسلة متقاربة يجب أن يكون $|2z| < 1$
 هذا هو المجال الذي نبحث عنه

$$= 1 - \sum_{m=0}^{\infty} (2z)^m$$

$$= 1 - \frac{1}{1 - 2z} \quad |2z| < 1$$

$$Y(z) = \frac{1-2z^{-1}}{1-2z} \quad \div (-2z)$$

(6)

$$Y(z) = \frac{z}{z - 1/2}$$

$$|z| < 1/2$$

Both transform of $x[n]$ and $y[n]$ have the same transform but different Roc.

Example 2

$$x[n] = \left(\frac{1}{2}\right)^n u[n] + \left(\frac{1}{3}\right)^n u[n]$$

find $X(z)$

$$X(z) = \frac{z}{z - 1/2} + \frac{z}{z - 1/3}$$

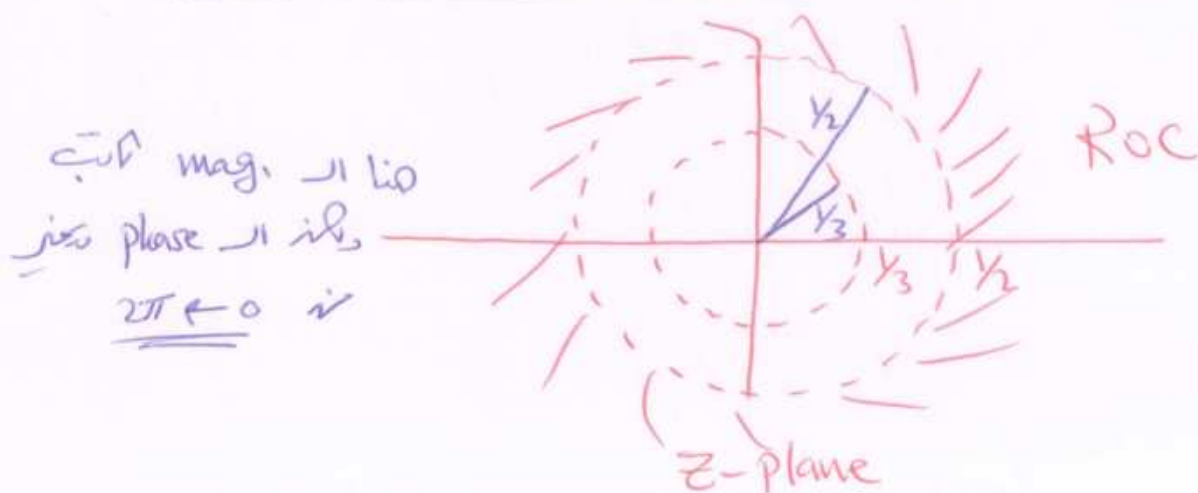
$$|z| > 1/2$$

في الخارج

$$|z| > 1/3 \Rightarrow \max(1/2, 1/3) = 1/2$$

في الخارج

$$\therefore X(z) = \frac{z}{z - 1/2} + \frac{z}{z - 1/3} \quad \& \quad \text{Roc } |z| > 1/2$$



⊛ Relationship between ZT & LT

[7]

in LT
$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

let $t = nT$
$$= \lim_{T \rightarrow 0} \sum_{n=-\infty}^{\infty} x(nT) (e^{sT})^n \cdot T$$

$$X(z) = \lim_{T \rightarrow 0} \sum_{n=-\infty}^{\infty} X[n] (e^{sT})^n \cdot T$$

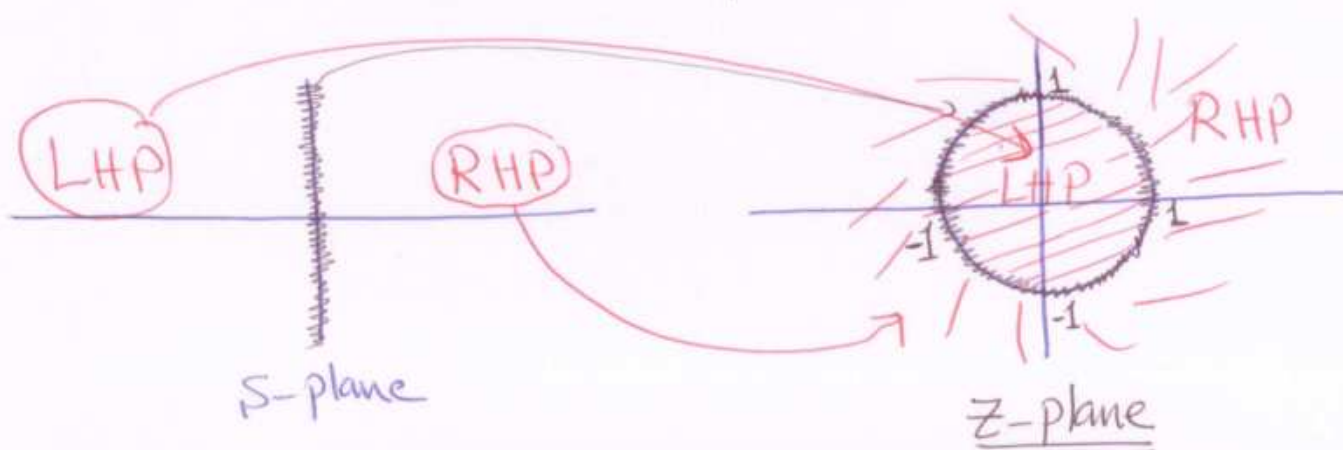
Z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} X[n] z^{-n}$$

Can think of z-transform a DT version of Laplace transform with

$$z = e^{sT} \quad (**)$$

- jw-axis in s-plane ($s = jw$) $\Leftrightarrow |z| = |e^{jwT}| = 1$
it is a unit circle in z-plane



- LHP in s-plane, $\text{Re}\{s\} < 0 \Rightarrow |z| = |e^{j\omega T}| < 1$ that means inside the $|z|=1$ circle
- RHP in s-plane, $\text{Re}\{s\} > 0 \Rightarrow |z| = |e^{j\omega T}| > 1$, outside the $|z|=1$ circle
- A vertical line in s-plane, $\text{Re}\{s\} = \text{constant} \Rightarrow |e^{j\omega T}| = \text{constant}$, a circle in z-plane

i.e.

(*) Some properties of z-transform

[1] linearity

if

$$\begin{aligned} x_1[n] &\longleftrightarrow X_1(z) \\ x_2[n] &\longleftrightarrow X_2(z) \end{aligned}$$

then

$$a x_1[n] + b x_2[n] \longleftrightarrow a X_1(z) + b X_2(z)$$

[2] time shifting a) Right shift of $x[n] \rightarrow$ (delay)

if

$$x[n] \longleftrightarrow X(z)$$

then

$$Z\{x[n-n_0]\} = z^{-n_0} X(z) + X[-n_0] + z^{-1} X[-n_0+1] + \dots + z^{-n_0+1} X[-1]$$

trick: for $n_0=1$, $X[-1]$ is the value of $x[n]$ at $n=-1$

i.e.

$$Z[x[n-1]] = z^{-1} X(z) + X[-1]$$

H.

Note that if $x[n]=0$ for $n=-1, -2, \dots, -n_0$ then

$$Z[x[n-n_0]] = z^{-n_0} X(z) \text{ (only)}$$

i.e $\mathcal{Z}\{x[n-2]\} = z^{-2}X(z) + \underbrace{x(-2) + z^{-1}x(-1)}_{\text{initial conditions}}$ #

b) Left shift in time $\rightarrow$ (Advanced)

$$x[n+n_0] \longleftrightarrow z^{n_0}X(z) - x(0)z^{n_0} - x(1)z^{n_0-1} - \dots - x[n_0-1]z$$

$x[n_0-1] \cdot z$

i.e $\mathcal{Z}\{x[n+2]\}$

$$= z^2X(z) - x(0)z^2 - x(1)z^1$$

#

Ex 2

$$y[n] - \frac{1}{2}y[n-1] = \delta[n], \quad y(-1) = 3$$

find $y[n]$

$$Y(z) - \frac{1}{2} [z^{-1}Y(z) + y(-1)] = 1$$

$$Y(z) - \frac{1}{2} z^{-1}Y(z) - \frac{3}{2} = 1$$

$$Y(z) \left[1 - \frac{1}{2z} \right] = 1 + \frac{3}{2}$$

$$Y(z) = \frac{5/2}{1 - \frac{1}{2z}} \cdot \frac{z}{z}$$

$$Y(z) = \frac{5/2 z}{z - 1/2} \stackrel{\text{OR}}{=} \frac{5/2}{1 - 1/2 z^{-1}}$$

$$y[n] = \frac{5}{2} \left(\frac{1}{2} \right)^n u[n]$$

#

Ex1 Solve the Difference equ.

101

$$y[n+2] - y[n+1] + \frac{2}{9} y[n] = x[n]$$

for $y[n]$, $n \geq 0$ if $x[n] = u[n]$ & $y[1] = 1$, $y[0] = 1$

$$[z^2 Y(z) - Y(0)z^2 - Y(1)z] - [zY(z) - Y(0)z] + \frac{2}{9} Y(z) = \frac{z}{z-1}$$

$$z^2 Y(z) - z^2 - z - zY(z) + z + \frac{2}{9} Y(z) = \frac{z}{z-1}$$

$$Y(z) (z^2 - z + \frac{2}{9}) = \frac{z}{z-1} + z^2$$

$$Y(z) (z^2 - z + \frac{2}{9}) = \frac{z + z^3 - z^2}{(z-1)}$$

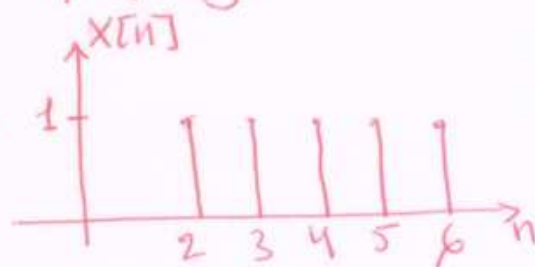
$$Y(z) = \frac{z^3 - z^2 + z}{(z-1)(z^2 - z + \frac{2}{9})}$$

$$\therefore y[n] = \mathcal{Z}^{-1} \{ Y(z) \}$$

L.

Ex2 Find $X(z)$ by using right shift property

$$x[n] = u[n-2] - u[n-7]$$



$$X(z) = z^{-2} X'(z) - z^{-7} X'(z)$$

$$= z^{-2} \cdot \frac{z}{z-1} - z^{-7} \cdot \frac{z}{z-1}$$

$$X(z) = \left(z^{-2} - z^{-7} \right) \cdot \frac{z}{z-1}$$

where All initial condition zero

$$X'(z) = \frac{z}{z-1}$$

وكون $u[n]$ 1

[3] Frequency Scaling (Multiplication by a^n)

III

if $x[n] \longleftrightarrow X(z)$

then $a^n x[n] \longleftrightarrow X\left[\frac{z}{a}\right]$

Ex2 $y[n] = (a^n \cos \Omega_0 n) u[n]$

$\Omega_0 = \omega_0 T$

from table $Z \{ \cos(\Omega_0 n) u[n] \} = \frac{z^2 - z \cos \Omega_0}{z^2 - 2z \cos \Omega_0 + 1}$

then

$$Y(z) = \frac{\left(\frac{z}{a}\right)^2 - \frac{z}{a} \cdot \cos(\Omega_0)}{\left(\frac{z}{a}\right)^2 - 2 \frac{z}{a} \cos(\Omega_0) + 1} \quad \#$$

[4] Differentiation with respect to z

if $x[n] \longleftrightarrow X(z)$

then $n^k x[n] \longleftrightarrow \left(-z \frac{d}{dz}\right)^k X(z)$

Ex2 $y[n] = n(n+1) u[n]$ find $Y(z)$

$$y[n] = n^2 u[n] + n u[n]$$

$$\begin{aligned} Z \{ n u[n] \} &= \left(-z \frac{d}{dz}\right) \left(\frac{z}{1-z}\right) = -z \cdot \left(\frac{1(z-1) - z \cdot 1}{(z-1)^2}\right) \\ &= -z \cdot \left(\frac{-1}{(z-1)^2}\right) \end{aligned}$$

$$= \frac{z}{(z-1)^2}$$

$$Z \{ n^2 x[n] \} = \left(-z \frac{d}{dz} \right)^2 \cdot \frac{z}{(z-1)}$$

$$= -z \frac{d}{dz} \cdot \frac{z}{(z-1)^2}$$

$$= -z \left[\frac{(z-1)^2 - z(2(z-1)) \cdot 1}{((z-1)^2)^2} \right]$$

$$= -z \left[\frac{(z-1)(z-1-2z)}{(z-1)^4} \right]$$

$$= \frac{z(z+1)}{(z-1)^3}$$

$$\therefore Y(z) = \frac{z(z+1)}{(z-1)^3} + \frac{z}{(z-1)^2} = \frac{2z^2}{(z-1)^3} \quad \text{P.}$$

[5] Time reversal

if

$$x[n] \longleftrightarrow X(z)$$

then

$$x[-n] \longleftrightarrow X\left(\frac{1}{z}\right)$$

[6] Initial value Thm 2

[13]

$$x(0) = \lim_{n \rightarrow 0} x[n] = \lim_{z \rightarrow \infty} X(z)$$

find initial value
of CT in Z-Domain
time Domain &
Z-Domain

Ex 2

$$\frac{X(z)}{z} = \frac{z^3 - \frac{3}{4}z^2 + 2z - \frac{5}{4}}{(z-1)(z-\frac{1}{3})(z^2 - \frac{1}{2}z + 1)}$$

find $x(0)$?

$$x(0) = \lim_{z \rightarrow \infty} \frac{z^4 - \frac{3}{4}z^3 + 2z^2 - \frac{5}{4}z}{(z-1)(z-\frac{1}{3})(z^2 - \frac{1}{2}z + 1)}$$

$$= \lim_{z \rightarrow \infty} \frac{z^4 - \frac{3}{4}z^3 + 2z^2 - \frac{5}{4}z}{z^4 - \frac{11}{6}z^3 + 2z^2 - \frac{9}{6}z + \frac{1}{3}}$$

✓ 1st 4 terms
cancel out

$$= \lim_{z \rightarrow \infty} \frac{1 - \frac{3}{4}z^{-1} + 2z^{-2} - \frac{5}{4}z^{-3}}{1 - \frac{11}{6}z^{-1} + 2z^{-2} - \frac{9}{6}z^{-3} + \frac{1}{3}z^{-4}}$$

$$x(0) = \boxed{1}$$

✓

[7] Final value Thm 2

$$x(\infty) = \lim_{n \rightarrow \infty} x[n] = \lim_{z \rightarrow 1} X(z) (z-1) = \lim_{z \rightarrow 1} X(z) (1-z^{-1})$$

The system must be stable system to be has final value

Before find it, test the stability as in CT
by using mapping method.

Ex1

$$X(z) = \frac{z^3 - \frac{3}{4}z^2 + 2z - \frac{5}{4}}{(z-1)(z-\frac{1}{3})(z^2 - \frac{1}{2}z + 1)}$$

find $X(\infty)$

- First step, test the stability (To be stable All poles must be inside the unit circle) in z-plane
- find the characteristic equation

$$(z-1)(z-\frac{1}{3})(z^2 - \frac{1}{2}z + 1) = 0$$

$$(z-1)(z-\frac{1}{3})(z-0.25-j0.96)(z+0.25+j0.96)$$

- All poles inside the unit circle that means it is stable system.

$$|z| = 0.992$$

inside the unit circle

$$X(\infty) = \lim_{z \rightarrow \infty} (z-1)X(z) = \lim_{z \rightarrow \infty} \frac{z^3 - \frac{3}{4}z^2 + 2z - \frac{5}{4}}{(z-\frac{1}{3})(z^2 - \frac{1}{2}z + 1)}$$

$$X(\infty) = \boxed{1}$$

$$\boxed{|z|=1} \text{ is inside}$$

Ex2

$$x[n] = 2^n u[n]$$

find $X(\infty)$

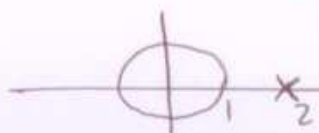
$$X(\infty) = 2^\infty u[\infty] = \infty$$

unstable system
there is no final value

also in z-plane

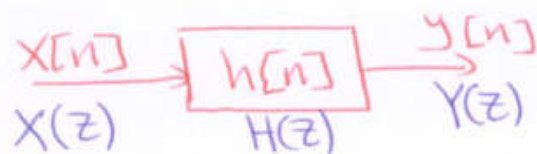
$$X(z) = \frac{z}{z-2}$$

$$|z| = 2 \text{ outside of unit circle.}$$



[4] Convolution 2

[15]



$$y[n] = x[n] * h[n]$$

$$Y(z) = X(z) \cdot H(z)$$

Convolution in time means
multiplication in frequency

Ex 2

given $h[n] = \{1, 2, 0, -1, 1\}$ & $x[n] = \{1, 3, -1, -2\}$

find $y[n]$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$H(z) = 1 + 2z^{-1} + 0z^{-2} - z^{-3} + z^{-4}$$

$$X(z) = 1 + 3z^{-1} - z^{-2} - 2z^{-3}$$

then

$$Y(z) = X(z) \cdot H(z)$$

$$X(z) = 1 + 3z^{-1} - z^{-2} - 2z^{-3}$$

$$H(z) = 1 + 2z^{-1} - z^{-3} + z^{-4}$$

$$1 + 3z^{-1} - z^{-2} - 2z^{-3}$$

$$0 + 2z^{-1} + 6z^{-2} - 2z^{-3} - 4z^{-4}$$

$$0 + 0 - z^{-3} - 3z^{-4} + z^{-5} + 2z^{-6}$$

$$0 + 0 + 0 + z^{-4} + 3z^{-5} - z^{-6} - 2z^{-7}$$

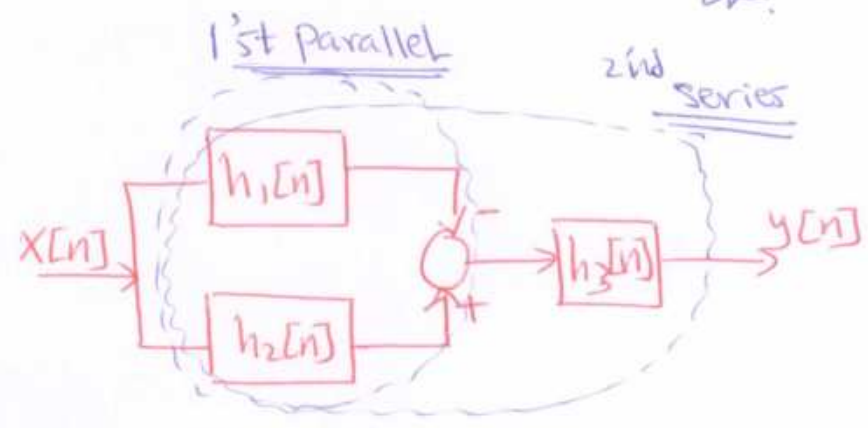
$$1 + 5z^{-1} + 5z^{-2} - z^{-3} - 4z^{-4} - 2z^{-5} - 4z^{-6} + 4z^{-7} - 2z^{-8}$$

$$Y(z) = 1 + 5z^{-1} + 5z^{-2} - 5z^{-3} - 6z^{-4} + 4z^{-5} + z^{-6} - 2z^{-7}$$

$$y[n] = \{ \underset{\uparrow}{1}, 5, 5, -5, -6, 4, 1, -2 \}$$

2B, 128:
Ex 2

given 2



$$h_1[n] = (n-1)u[n]$$

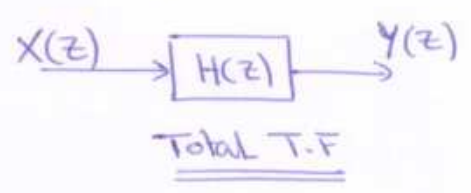
$$h_2[n] = \delta[n] + n u[n-1] + \delta[n-2]$$

$$h_3[n] = \left(\frac{1}{2}\right)^n u[n]$$

we need first to reduced the block diagram to single one as

find T.F?

$$H(z) = \frac{Y(z)}{X(z)}$$



$$H(z) = [H_2(z) - H_1(z)] \cdot H_3(z)$$

Now from table find the z-transform for sub systems.

$$\hookrightarrow h_1[n] = n u[n] - u[n]$$

$$\begin{aligned} H_1(z) &= \left(-z \frac{d}{dz}\right)^1 X(z) - \frac{z}{(z-1)} \quad \left\{ \begin{array}{l} \text{another way} \\ \frac{d}{dz} \left(\frac{z}{(z-1)} \right) = \\ \frac{d}{dz} (z \cdot (z-1)^{-1}) \end{array} \right. \\ &= -z \cdot \frac{(z-1) - z \cdot 1}{(z-1)^2} - \frac{z}{(z-1)} \\ &= \frac{z}{(z-1)^2} - \frac{z}{(z-1)} = \frac{2z - z^2}{(z-1)^2} \end{aligned}$$

$$H_1(z) = \frac{2z - z^2}{(z-1)^2}$$

$$\hookrightarrow h_2[n] = \delta[n] + n u[n-1] + \delta[n-2]$$

$$u[n] \leftrightarrow \frac{z}{z-1}$$

$$u[n-1] \leftrightarrow z^{-1} \cdot \frac{z}{z-1} = \frac{1}{(z-1)}$$

$$n u[n-1] \leftrightarrow \left(-z \frac{d}{dz} \right)^1 \left(\frac{1}{(z-1)} \right) = \frac{0 - 1 \cdot (-z)}{(z-1)^2}$$

$$= \frac{z}{(z-1)^2}$$

$$\therefore H_2(z) = 1 + \frac{z}{(z-1)^2} + z^{-2} \cdot 1$$

$$= \frac{(z^{-2}+1)(z-1)^2 + z}{(z-1)^2} = \frac{((z^{-2}+1)(z^2-2z+1)) + z}{(z-1)^2}$$

$$= \frac{1 - 2z^{-1} + z^{-2} + z^2 - 2z + 1 + z}{(z-1)^2} \cdot \frac{z^{+2}}{z^{+2}}$$

$$= \frac{z^4 - z^3 + 2z^2 - 2z + 1}{z^2(z-1)^2}$$

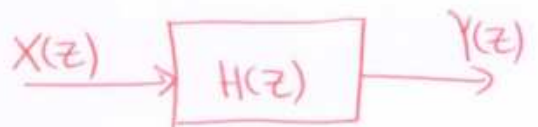
$$\hookrightarrow H_3(z) = \frac{z}{z-1/2}$$

$$\therefore H(z) = H_2(z) H_3(z) = H_1(z) H_3(z)$$

T.F لا يوجد total في $H(z)$ لا يوجد في $H(z)$ #

⊗ Discrete time Transfer function 2

- All Initial condition equal Zero



$$\boxed{T.F = H(z) = \frac{Y(z)}{X(z)}}$$

• From Difference equ. to T.F 2

Ex: Delay form

$$2y[n] + 3y[n-1] + 5y[n-2] = x[n-1] - 2x[n-2]$$

find T.F , given $y[-1]=2, y[-2]=3$
I.C = zero

$$2Y(z) + 3z^{-1}Y(z) + 5z^{-2}Y(z) = z^{-1}X(z) - 2z^{-2}X(z)$$

$$Y(z) (2 + 3z^{-1} + 5z^{-2}) = X(z) (z^{-1} - 2z^{-2})$$

$$Y(z) = \frac{z^{-1} - 2z^{-2}}{2 + 3z^{-1} + 5z^{-2}} X(z)$$

↳ zero-state response

- the zero input response = zero because All I.C = zero

$$\therefore \frac{Y(z)}{X(z)} = \frac{z - 2}{2z^2 + 3z + 5}$$

Ans

Ex.

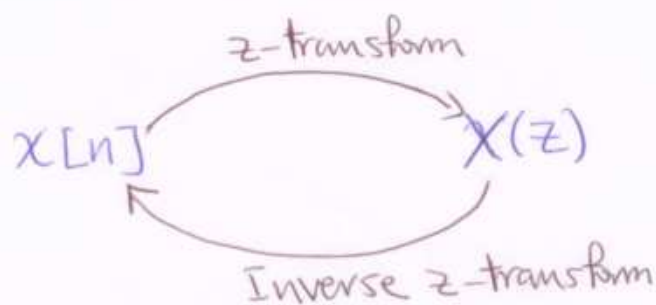
$$2y[n+2] + 3y[n+1] + 5y[n] = x[n+1] - 2x[n]$$

find T.F.

$$2z^2 Y(z) + 3z Y(z) + 5 Y(z) = z X(z) - 2 X(z)$$

$$\frac{Y(z)}{X(z)} = \frac{z - 2}{2z^2 + 3z + 5} \quad \#$$

⊗ Inverse of z-transform



→ There are 3 methods to find the inverse z-transform

[1] Direct division method (power series method) (power series method)

[2] by partial fraction expansion

[3] by Integral method.

I] Direct Division Method

Ex 1

$$X(z) = \frac{z}{z - 0.1}$$

$$|z| > 0.1$$

find $x[n]$

$$\begin{array}{r} 1 + (0.1)z^{-1} + (0.1)^2 z^{-2} + \dots \\ z - 0.1 \overline{) z} \end{array}$$

$$\therefore X(z) = 1 + (0.1)z^{-1} + (0.1)^2 z^{-2} + (0.1)^3 z^{-3} + \dots$$

index $\begin{matrix} n \\ 0 \quad n=1 \quad n=2 \quad n=3 \quad \dots \end{matrix}$

$$\begin{array}{r} + z \pm 0.1 \\ 0.1 \\ + 0.1 \pm (0.1)^2 z^{-1} \\ (0.1)^2 z^{-1} \\ + (0.1)^2 z^{-1} \pm (0.1)^3 z^{-2} \\ (0.1)^3 z^{-2} \\ \vdots \end{array}$$

$$\therefore x[n] = (0.1)^n \quad n \geq 0$$

$$\therefore x[n] = (0.1)^n u[n] \leftarrow (\text{By table})$$

$$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$$

هذه الطريقة صالحة في كثير من الحالات خاصة إذا كانت $x[n]$ بسيطة

II] Inversion by partial fraction expansion

أولاً نأخذ شكل $X(z)$ ونكتبه على صورة كسور جزئية $X(z) = \frac{4z}{z - 1/2}$

إذا كان لدينا $\frac{4}{z - 1/2}$ P.F. في $z = 1/2$

نوجد K في الشكل $\frac{4z}{z - 1/2}$ عند $z = 1/2$

بإيجاد K في الشكل $\frac{4z}{z - 1/2}$ عند $z = 1/2$

Example 2

(21)

$$X(z) = \frac{z^3 - z^2 + z - 1/16}{z^3 - 5/4 z^2 + 1/2 z - 1/16}$$

$$|z| > 1/2$$

find $x[n]$?

z دال دى بى آسم دى

$$\frac{X(z)}{z} = \frac{z^3 - z^2 + z - 1/16}{z(z^3 - 5/4 z^2 + 1/2 z - 1/16)}$$

$$\frac{X(z)}{z} = \frac{z^3 - z^2 + z - 1/16}{z(z - 1/2)^2(z - 1/4)}$$

By using
Partial
Fraction

$$\frac{X(z)}{z} = \frac{k_1}{z} + \frac{k_2}{(z - 1/2)^2} + \frac{k_3}{(z - 1/2)} + \frac{k_4}{(z - 1/4)}$$

$$k_1 = z \frac{X(z)}{z} \Big|_{z=0} = \frac{z^3 - z^2 + z - 1/16}{(z - 1/2)^2(z - 1/4)} \Big|_{z=0} = \boxed{1}$$

$$k_2 = \frac{z^3 - z^2 + z - 1/16}{z(z - 1/4)} \Big|_{z=1/2} = \boxed{5/2}$$

$$k_3 = \frac{1}{1!} \frac{d}{dz} \left[\frac{z^3 - z^2 + z - 1/16}{z(z - 1/4)} \right] \Big|_{z=1/2} = \frac{(3z^2 - 2z + 1)(z^2 - 1/4z) - (z^3 - z^2 + z - 1/16)}{(z^2 - 1/4z)^2} \Big|_{z=1/2}$$

$$k_3 = \boxed{-9}$$

$$k_4 = \frac{z^3 - z^2 + z - 1/16}{z(z - 1/2)^2} \Big|_{z=1/4} = \boxed{9}$$

$$\frac{X(z)}{z} = \frac{1}{z} + \frac{5/2}{(z - 1/2)^2} + \frac{-9}{(z - 1/2)} + \frac{9}{(z - 1/4)}$$

$$X(z) = 1 + \frac{5/2 z}{(z - 1/2)^2} - \frac{9z}{(z - 1/2)} + \frac{9z}{z - 1/4}$$

By using table 2

$$X[n] = \delta[n] + \left[\frac{5}{2} n \left(\frac{1}{2}\right)^n - 9 \left(\frac{1}{2}\right)^n + 9 \left(\frac{1}{4}\right)^n \right] u[n]$$

Example(2):

$$X(z) = \frac{2z(3z + 17)}{(z - 1)(z^2 - 6z + 25)}$$

z poles ①

$$\frac{X(z)}{z} = \frac{2(3z + 17)}{(z - 1)(z - 3 - j4)(z - 3 + j4)}$$

$$\frac{X(z)}{z} = \frac{k_1}{(z - 1)} + \frac{k_2}{(z - 3 - j4)} + \frac{k_2^*}{(z - 3 + j4)}$$

$$k_1 = \frac{2(3z + 17)}{z^2 - 6z + 25} \Big|_{z=1} = \underline{2}$$

$$k_2 = \frac{2(3z + 17)}{(z - 1)(z - 3 + j4)} \Big|_{z=3+j4} = 1.6 \angle -128.65^\circ$$

degree

$$k_2^* = 1.6 \angle +128.65^\circ$$

$$\therefore X(z) = \frac{2z}{(z - 1)} + \frac{1.6 \angle -128.65^\circ}{z - 3 - j4} + \frac{1.6 \angle +128.65^\circ}{z - 3 + j4}$$

By using table & this formula ↓

Formula

$$\mathcal{Z}^{-1} \left\{ \frac{(0.5r e^{j\theta})z}{z-\gamma} + \frac{(0.5r e^{-j\theta})z}{z-\gamma^*} \right\} =$$

$$= r |\gamma|^n \cos(\beta n + \theta) u[n]$$

where 2

$$\gamma = |\gamma| \angle \theta = |\gamma| e^{j\beta}$$

i.e if $\gamma = 3 + j4 = 5 \angle 53.13 \Rightarrow |\gamma| = 5$
 $\beta = 53.13$

if $0.5r = 1.6 \Rightarrow r = 3.2$

by table & formula

$$x[n] = [2(1)^n + 3.2 * (5)^n \cos(53.13n - 128.65)] u[n]$$

↳ convert to radian

$$x[n] = [2(1)^n + 3.2(5)^n \cos(0.927n - 2.246)] u[n]$$

~~the~~

Ex 3

find $y[n]$ if $Y(z) = \frac{z+1}{z^2 + \frac{3}{4}z + \frac{1}{8}}$

$$\frac{Y(z)}{z} = \frac{(z+1)}{z(z+\frac{1}{2})(z+\frac{1}{4})} = \frac{k_1}{z} + \frac{k_2}{z+\frac{1}{2}} + \frac{k_3}{z+\frac{1}{4}}$$

$$Y(z) = \frac{8z}{z} + \frac{4z}{z+\frac{1}{2}} - \frac{12z}{z+\frac{1}{4}}$$

$$y[n] = 8\delta[n] + [4(-\frac{1}{2})^n - 12(-\frac{1}{4})^n] u[n]$$

#

Ex 2

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 2 \sin \frac{\pi}{2}n$$

given $y[-1] = 2$, $y[-2] = 4$ Solve the Diff eq.

Delay form (Right shift property)

$$Y(z) - \frac{3}{4} [\underbrace{\bar{z}^{-1} Y(z)}_{\text{delay}} \oplus \overset{2}{Y(-1)}] + \frac{1}{8} [\bar{z}^{-2} Y(z) \oplus \overset{4}{Y(-2)} \oplus \bar{z}^{-1} \overset{2}{Y(-1)}] = \frac{2z \sin \frac{\pi}{2}}{z^2 - 2z \cos \frac{\pi}{2} + 1}$$

$$Y(z) \left(1 - \frac{3}{4}\bar{z}^{-1} + \frac{1}{8}\bar{z}^{-2} \right) \mp \frac{3}{2} + \frac{1}{2} \mp \frac{1}{4}\bar{z}^{-1} = \frac{2z}{z^2 + 1}$$

$$Y(z) \left(1 - \frac{3}{4}\bar{z}^{-1} + \frac{1}{8}\bar{z}^{-2} \right) = \frac{2z}{z^2 + 1} + 1 + \frac{1}{4}\bar{z}^{-1} \quad \times \bar{z}^2 \text{ in spg}$$

$$\begin{aligned} Y(z) \left(z^2 - \frac{3}{4}z + \frac{1}{8} \right) &= \frac{2z^3}{z^2 + 1} + z^2 + \frac{1}{4}z \\ &= \frac{(z^2 + 1)z^2 + 2z^3}{(z^2 + 1)} + \frac{1}{4} \frac{z}{1} \\ &= (z^2 + 1)z^2 + 2z^3 + \frac{1}{4}z(z^2 + 1) \end{aligned}$$

$$\therefore Y(z) = \frac{z^4 + \frac{9}{4}z^3 + z^2 + \frac{1}{4}z}{z^2 - \frac{3}{4}z + \frac{1}{8}}$$

$$\frac{Y(z)}{z} = \frac{z^3 + \frac{9}{4}z^2 + z + \frac{1}{4}}{(z - 0.5)(z - 0.25)}$$

$$= \frac{k_1}{(z - \frac{1}{2})} + \frac{k_2}{z - \frac{1}{4}}$$

$$k_1 = \left. \frac{z^3 + \frac{9}{4}z^2 + z + \frac{1}{4}}{(z - \frac{1}{4})} \right|_{z=\frac{1}{2}} = 5.75$$

$$k_2 = \left. \frac{z^3 + \frac{9}{4}z^2 + z + \frac{1}{4}}{(z - 0.5)} \right|_{z=0.25} = -2.625$$

$$\therefore y[n] = \left[5.75 \left(\frac{1}{2}\right)^n - 2.625 \left(\frac{1}{4}\right)^n \right] u[n]$$

Ex 2

$$y[n] + y[n-1] - 2y[n-2] = 2x[n] - x[n-1]$$

find an input $x[n]$ with $x[n] = 0$ for $n < 0$

$$\& y[n] = 2[u[n] - u[n-3]] \quad \& y[-2] = 2, y[-1] = 0$$

means I.C = 0

$$Y(z) + [\bar{z}^{-1}Y(z) + \overset{0}{\cancel{Y(z)}}] - 2[\bar{z}^{-2}Y(z) + \overset{2}{Y(z)} + \bar{z}^{-1}\overset{0}{\cancel{Y(z)}}] = 2X(z) - \bar{z}^{-1}X(z)$$

$$Y(z)(1 + \bar{z}^{-1} - 2\bar{z}^{-2}) - 4 = (2 - \bar{z}^{-1})X(z)$$

$$\downarrow \text{as } Y(z) = 2 \left[\frac{z}{(z-1)} - \bar{z}^{-3} \cdot \frac{z}{z-1} \right] = \frac{2z - 2\bar{z}^{-2}}{(z-1)}$$

we will get this

$$\frac{2z - 2\bar{z}^{-1}}{(z-1)} (1 + \bar{z}^{-1} - 2\bar{z}^{-2}) - 4 = (2 - \bar{z}^{-1})X(z)$$

$$\frac{2z + 2 - 4z^{-1} - 2z^{-1} - 2z^{-2} + 4z^{-3}}{(z-1)} - 4 = (2 - z^{-1})X(z) \quad [26]$$

$$\frac{2z + 2 - 6z^{-1} - 2z^{-2} + 4z^{-3} - 4z + 4}{(z-1)} = (2 - z^{-1})X(z)$$

$$X(z) = \frac{-2z + 6 - 6z^{-1} - 2z^{-2} + 4z^{-3}}{(z-1)(2-z^{-1})} \cdot \frac{z^3}{z^3}$$

$$X(z) = \frac{-2z^4 + 6z^3 - 6z^2 - 2z + 4}{z^2(2z^2 - 3z + 1)}$$

$$\frac{X(z)}{z} = \frac{-2z^4 + 6z^3 - 6z^2 - 2z + 4}{z^3(z-1)(z-\frac{1}{2})}$$

$$= \frac{k_1}{z^3} + \frac{k_2}{z^2} + \frac{k_3}{z} + \frac{k_4}{(z-1)} + \frac{k_5}{(z-\frac{1}{2})}$$

$$k_1 = z^3 \times X(z) \Big|_{z=0} = \frac{4}{-1 \times -\frac{1}{2}} = [8]$$

$$k_2 =$$

الجزء الثابت

time domain من $X(z)$

Ex 2

$$X(z) = \frac{z(z^2 - 4z + 5)}{(z-3)(z-1)(z-2)}$$

find $x[n]$?

$$\frac{X(z)}{z} = \frac{(z^2 - 4z + 5)}{(z-3)(z-1)(z-2)}$$

$$= \frac{k_1}{(z-3)} + \frac{k_2}{(z-1)} + \frac{k_3}{(z-2)}$$

$$k_1 = \frac{z^2 - 4z + 5}{(z-1)(z-2)} \Big|_{z=3} = \boxed{1}$$

$$k_2 = \frac{z^2 - 4z + 5}{(z-3)(z-2)} \Big|_{z=1} = \boxed{1}$$

$$k_3 = \frac{z^2 - 4z + 5}{(z-3)(z-1)} \Big|_{z=2} = \boxed{-1}$$

$$\frac{X(z)}{z} = \frac{1}{(z-3)} + \frac{1}{(z-1)} + \frac{-1}{(z-2)}$$

$$X(z) = \frac{z}{(z-3)} + \frac{z}{(z-1)} - \frac{z}{(z-2)}$$

a) if ROC $|z| > 3$

$$\therefore x[n] = [(3)^n + (1)^n - (2)^n] u[n]$$

b) if ROC $|z| < 1$

All poles ve

[28]

$$X[n] = -(3)^n u[-n] - (1)^n u[-n] + (2)^n u(-n)$$

c) if ROC $2 < |z| < 3$

poles ① & ② positive poles ③ ve

$$X[n] = [-(3)^n u[-n] + (1)^n u[n] - (2)^n u[n]]$$

[3] Inversion integral method 2-

$$X[n] = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz$$

$$X[n] = \sum_{\substack{\text{at poles} \\ \text{of } [X(z) z^{n-1}]}} \left[\text{residue of } X(z) \cdot z^{n-1} \right]$$

inverse 1, 2, 3 & p.f. 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100

→ if the function $X(z) z^{n-1}$ has a simple poles at $z=a$ then the residue is not repeated

$$\text{Residue} \Big|_{z=a} = (z-a) X(z) z^{n-1} \Big|_{z=a}$$

→ for a pole of order m at $z=a$ the residue is

Repeated poles

$$\text{Residue} \Big|_{z=a} = \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left[(z-a)^m X(z) z^{n-1} \right] \Big|_{z=a}$$

Ex 1 Find $x[n]$ for $X(z) = \frac{z}{(z-1)^2}$

$$\frac{X(z)}{z} = \frac{1}{(z-1)^2}$$

$$= \frac{k_1}{(z-1)^2} + \frac{k_2}{(z-1)}$$

$$k_1 = \boxed{1}$$

$$k_2 = \frac{d}{dz}(1) = 0$$

$$X(z) = \frac{z}{(z-1)^2}$$

رصيد في الباقية ((الباقية))

... residue at $z=1$ is $x[n]$

CS 478

- First find $X(z) z^{n-1} = \frac{z}{(z-1)^2} \cdot z^{n-1}$

$$= \frac{z^n}{(z-1)^2}$$

poles at $\boxed{z=1}$ & repeats

So residue $\Big|_{z=1} = \frac{1}{(2-1)!} \frac{d}{dz} \left[\cancel{(z-1)^2} \cdot \frac{z^n}{\cancel{(z-1)^2}} \right] \Big|_{z=1}$

$$= \frac{d}{dz} [z^n] \Big|_{z=1} = n z^{n-1} \Big|_{z=1} = \boxed{n}$$

So $x[n] = n \quad u[n] \quad \neq$

Ex 2

30

$$X(z) = \frac{z^2}{(z-1)^2(z-e^{-aT})}$$

$$X[n] = \sum_{\substack{\text{at poles} \\ \text{of } X(z)z^{n-1}}} \left[\text{residue of } X(z)z^{n-1} \right]$$

$$= \sum_{i=1}^2 \left[\text{residue of } \frac{z^2 \cdot z^{n-1}}{(z-1)^2(z-e^{-aT})} \right] \Big|_{\text{at } z=z_i}$$

$$= k_1 + k_2$$

$$k_1 = \left[\text{residue at simple poles } z=e^{-aT} \right]$$

$$= \left(\cancel{z-e^{-aT}} \right) \cdot \frac{z^{n+1}}{(z-1)^2(\cancel{z-e^{-aT}})} \Big|_{z=e^{-aT}}$$

$$= \frac{e^{-a(n+1)T}}{(e^{-aT}-1)^2}$$

$$k_2 = \left[\text{residue at double poles } z=1 \right] \Big|_{z=1}$$

$$= \frac{1}{(2-1)!} \cdot \frac{d}{dz} \left[(\cancel{z-1})^2 \cdot \frac{z^{n+1}}{(\cancel{z-1})^2(z-e^{-aT})} \right] \Big|_{z=1}$$

$$= \frac{(n+1)z^n \cdot (z-e^{-aT}) - z^{n+1} \cdot 1}{(z-e^{-aT})^2} \Big|_{z=1}$$

$$k_2 = \left. \frac{(n+1)z^n}{(z - e^{-aT})} - \frac{z^{n+1}}{(z - e^{-aT})^2} \right|_{z=1}$$

$$= \frac{(n+1)}{(1 - e^{-aT})} - \frac{1}{(1 - e^{-aT})^2}$$

$$x[n] = k_1 + k_2$$

$$x[n] = \frac{e^{-a(n+1)T}}{(e^{-aT} - 1)^2} + \frac{n+1}{(1 - e^{-aT})} - \frac{1}{(1 - e^{-aT})^2}$$

Ex: find $x[n]$ if $X(z) = \frac{z^{-2}}{(1 - z^{-1})^3}$ $|z| > 1$ $\neq$

$$X(z) = \frac{z^{-2}}{(z-1)^3} = \frac{z^{-2}}{\frac{1}{z^3}(z-1)^3} = \frac{z}{(z-1)^3}$$

$$\therefore X(z) = \frac{z}{(z-1)^3} \Rightarrow X(z)z^{n-1} = \frac{z^n}{(z-1)^3}$$

$$x[n] = \left[\text{residue at triple poles } z=1 \right] \Big|_{z=1}$$

$$= \frac{1}{2!} \frac{d^2}{dz^2} \left[\cancel{(z-1)^3} \cdot \frac{z^n}{\cancel{(z-1)^3}} \right] \Big|_{z=1}$$

$$= \frac{1}{2!} \frac{d^2}{dz^2} [z^n] \Big|_{z=1} = \frac{1}{2} \frac{d}{dz} [n z^{(n-1)}]$$

$$= \frac{1}{2} n(n-1) z^{n-2} \Big|_{z=1}$$

$$x[n] = \left[\frac{1}{2} n(n-1) \right] u[n] \quad \neq$$

Ex 1

$$X(z) = \frac{z(z+2)}{(z-1)^2}$$

By Partial

$$\frac{X(z)}{z} = \frac{(z+2)}{(z-1)^2}$$

$$= \frac{k_1}{(z-1)^2} + \frac{k_2}{(z-1)}$$

$$k_1 = \underline{3} \quad , \quad k_2 = \underline{1}$$

$$\therefore X(z) = \frac{3z}{(z-1)^2} + \frac{z}{(z-1)}$$

$$x[n] = [3n(1)^n + (1)^n] u[n]$$

Another Solution

By inversion integral

$$x[n] = \frac{1}{(2-1)!} \frac{d}{dz} \left[\cancel{(z-1)^2} \cdot \left(\frac{z^n}{\cancel{(z-1)^2}} \right) \cdot (z+2) \right] \Big|_{z=1}$$

$x'(z) z^{n-1}$

$$= \frac{d}{dz} [z^n (z+2)] \Big|_{z=1}$$

$$= \frac{d}{dz} [z^{n+1} + 2z^n] \Big|_{z=1}$$

$$= (n+1)z^n + 2nz^{n-1}$$

$$= (n+1) + 2n = 3n+1$$

$$x[n] = [3n+1] u[n]$$

the same result