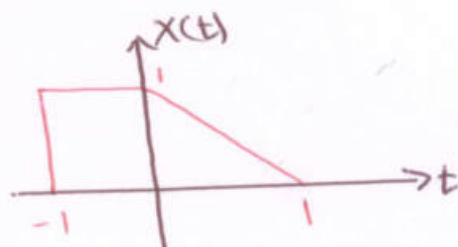


## chapter 6

Continuous time signal analysis  
"Fourier Series"⊗ Introduction 2

↳ As we studied before in chapter one how we can write the expression for any aperiodic signal in time domain by using useful function

$$x(t) = u(t+1) - r(t) + r(t-1)$$



↳ Also learned how we can use this Representation in time domain To find the Response for LTI system By using convolution operation which have this expression:

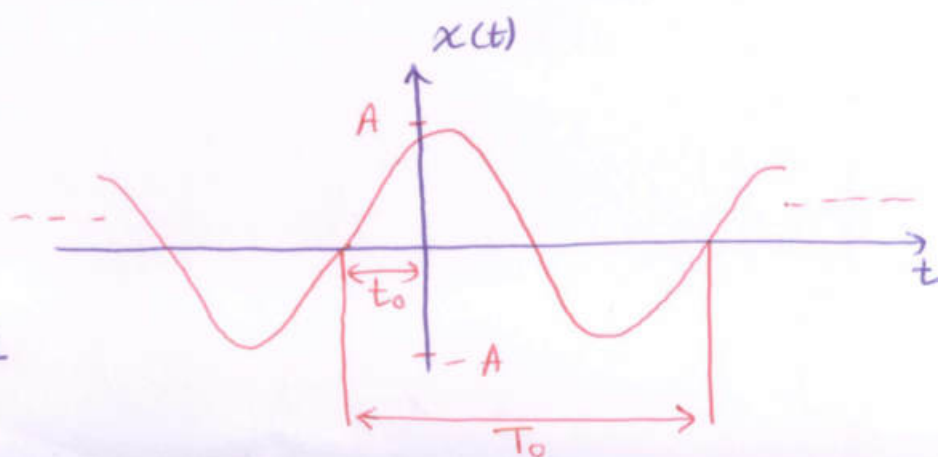
$$y(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau$$

for continuous signal

↳ Now, if you have periodic signal as the following Sinusoidal signal

- This signal called Sinusoidal signal

- and its periodic signal



- and it has general form or expression which is:

$$x(t) = A \sin(\omega_0 t + \phi)$$

where:

$A$  is the Amplitude of signal & it can be Volts or Amp

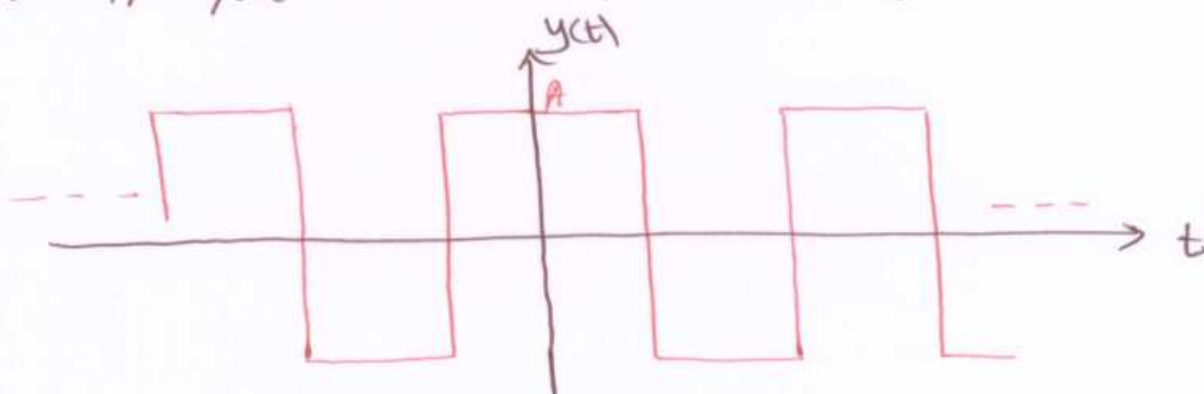
$\omega_0$  is called radian frequency (rad/sec)

$\phi$  is called the phase shift in radian.

$T_0$  is the period of sinusoid and its related to freq

$$\omega_0 = \frac{2\pi}{T_0}$$

↳ Now if you have this periodic signal

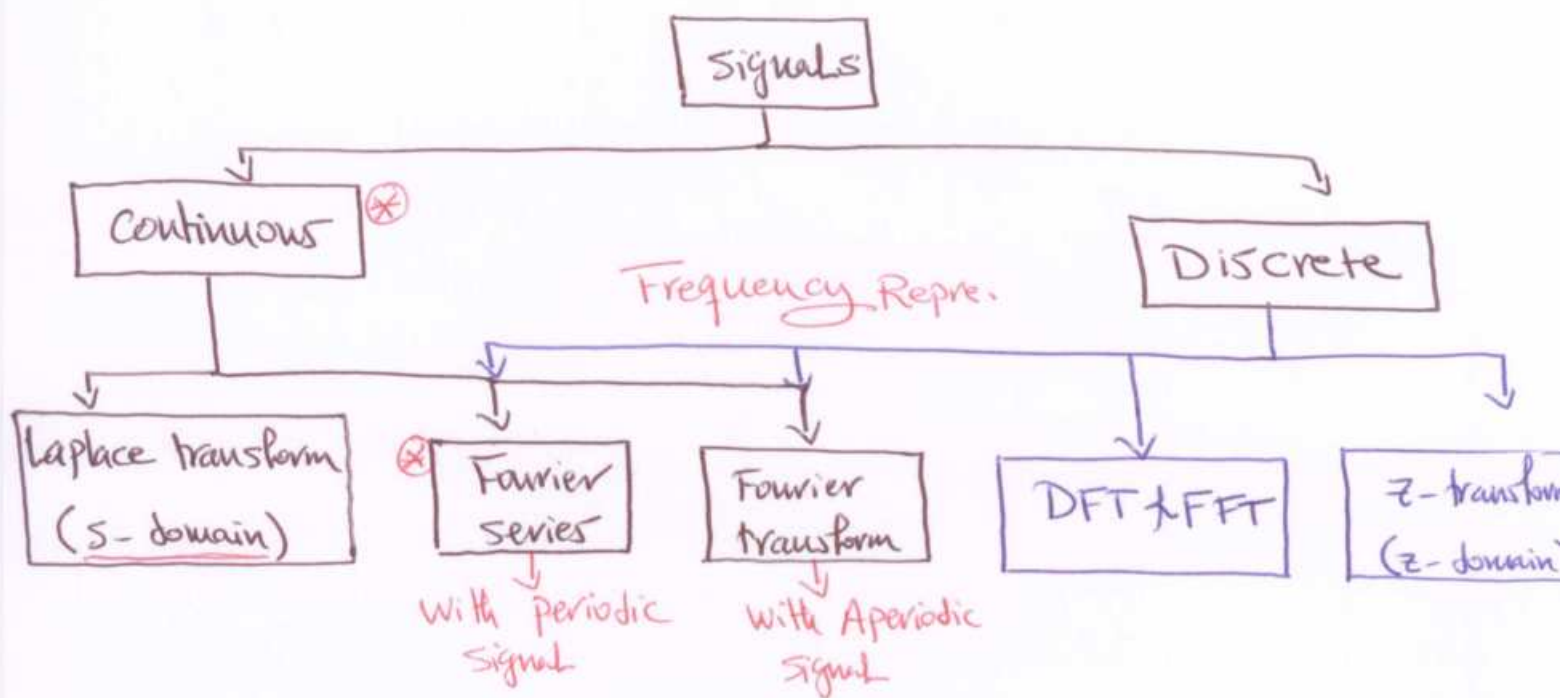


- This signal is not famous like sinusoidal and there is no general form for it, so I need to find another way to represent these type of signals.

↳ In this case I need to use Fourier analysis

## • Fourier analysis 2

its a type of Frequency Representation for signal as we see in the following chart.



• what is the Fourier series & how its works 2-

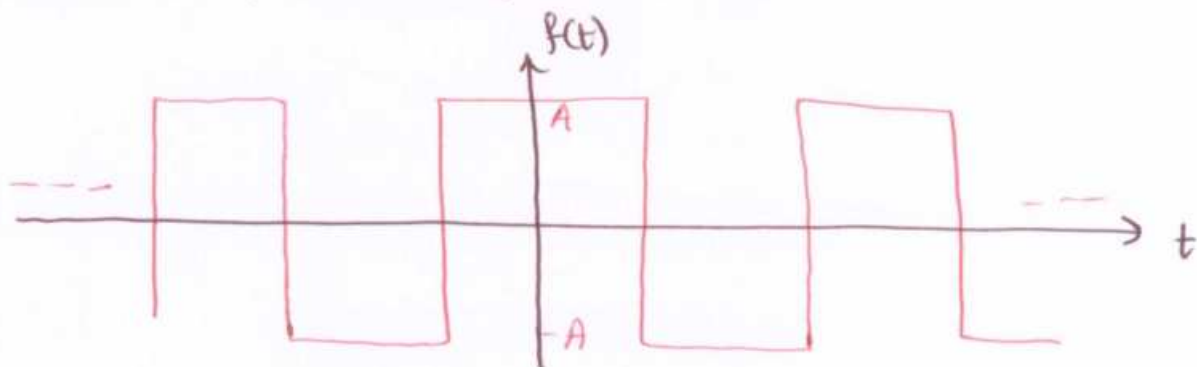
### • Fourier series 2-

It is a technique that represent any signal (periodic) as summation of sinusoidal ( $\sin x \cos$ ) or complex exponential.

and we will learn how we can use this representation to find the response of the system and how we can get the line spectra for CT signals.

## • how Fourier series works

- if you have periodic signal as  $f(t)$



- you can represent as combination of sinusoidal function as this expression.

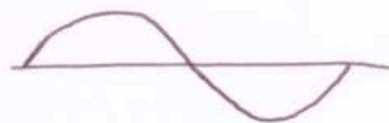
$$f(t) = a_0 + \sum_{n=1}^{\infty} [a_n \cos n\omega t + b_n \sin n\omega t]$$

where  $n$  called harmonic order.

### • harmonic

is a component frequency of the signal that is an ~~integer~~ integer multiply of the fundamental frequency

1st harmonic = fundamental signal with  $f$



2nd harmonic =  $2f$

$\vdots$

$\vdots$



### • what is the fundamental frequency?

- fundamental frequency is defined as the lowest frequency of periodic signal
- fundamental period is defined as the smallest period of periodic signal

• how we can find it?

e.g.

$$x(t) = \cos t/3 + \cos t/4$$

first you need to test if the given signal  $x(t)$  is periodic or not By:

$$T_1 = \frac{2\pi}{1/3} = 6\pi$$

$$T_2 = \frac{2\pi}{1/4} = 8\pi$$

$$\frac{T_1}{T_2} = \frac{6\pi}{8\pi} = \frac{6}{8} = \frac{3}{4}$$

Two periods must be equal ✓

This is rational number  
so the signal is periodic

⇒ the fundamental period  $T_0$  equal

$$T_0 = 3T_2 = 4T_1 = \underline{\underline{24\pi}} \text{ sec.}$$

$$f_0 = \frac{1}{24\pi} \text{ Hz}$$

Notes

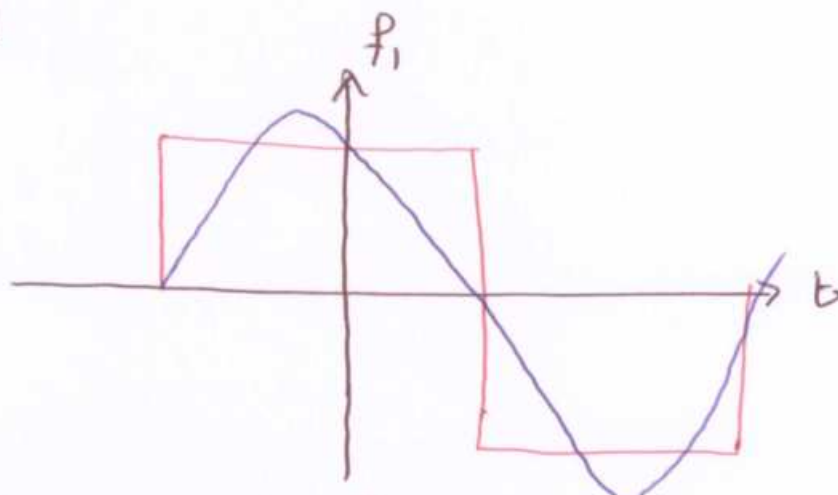
Any signal that satisfies this equation

$$x(t) = x(t + T_0) \quad (*)$$

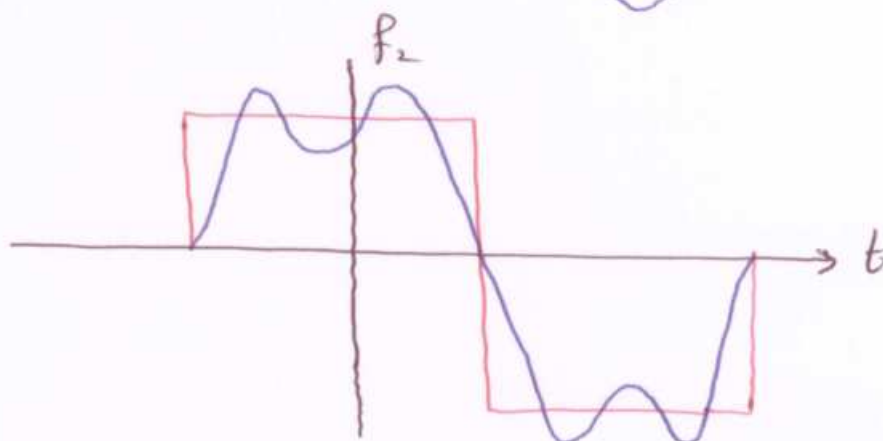
is called periodic signal

→ Now, back to how Fourier series works -  
take one period

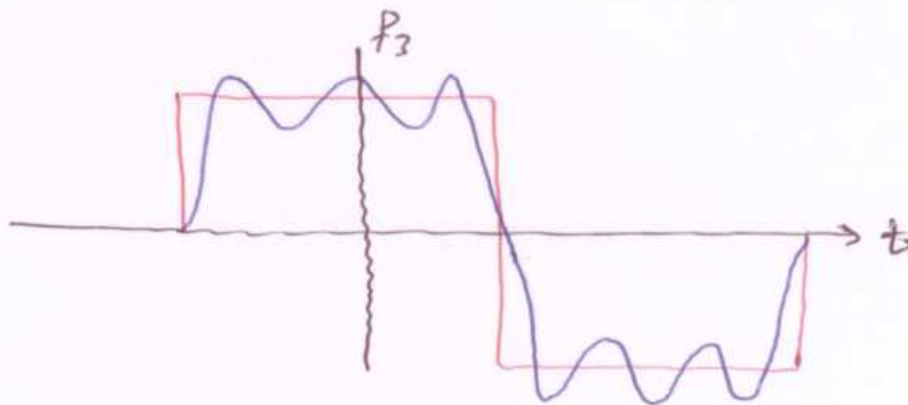
1st harmonic  
at  $n=1$



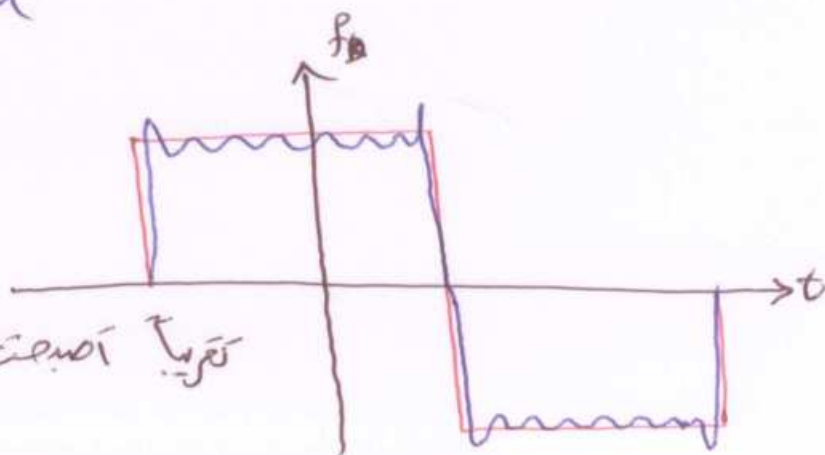
2nd harmonic  
at  $n=2$



3rd harmonic  
at  $n=3$



Now find the summation  
for  $n$ -harmonic then  
you get this function



approx as  $n \rightarrow \infty$  approx  $f_0$

## [1] Trigonometric Fourier series 2

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- It a method that convert the time domain model into frequency domain model

given time domain  
signal  $x(t)$



Find the FS coefficients  
 $\{a_0, a_n, b_n\}$

- Its expression 2

$$f(t) = a_0 + \sum_{n=1}^{\infty} [a_n \cos n\omega_0 t + b_n \sin n\omega_0 t]$$

where  $a_0, a_n, b_n$  called Fourier series coefficient 2

$$- a_0 = \frac{1}{T_0} \int_{0(T_0)}^{T_0} f(t) dt \quad (\text{DC - component})$$

$$- a_n = \frac{2}{T_0} \int_{0(T_0)}^{T_0} f(t) \cdot \cos(n\omega_0 t) dt$$
$$- b_n = \frac{2}{T_0} \int_{0(T_0)}^{T_0} f(t) \cdot \sin(n\omega_0 t) dt$$

} called Trigonometric Fourier series coefficient

Notes: the boundary for the previous integral mustn't be start from zero its the integral over one period ( $T_0$ )

e.g

$$a_0 = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2}$$

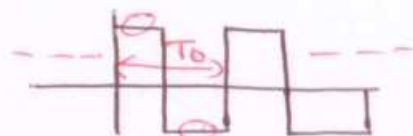
## \* Dirichlet condition - (To apply Fourier series)

1)  $f(t)$  periodic signal must be single-valued  
 $f(x,y) \rightsquigarrow \underline{f(t)}$   $\Leftarrow$  one dimension in  $t$   $\Leftarrow$

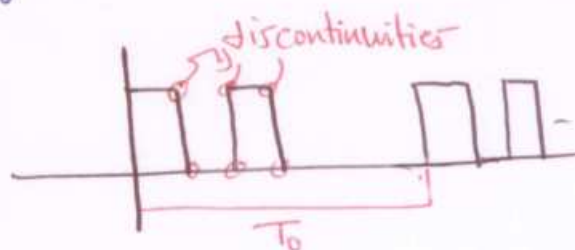
2) the absolute of  $x(t)$  integrable over one period & existence

$$\int_{T_0} |f(t)| dt < \infty$$

3) In a finite time interval,  $x(t)$  has a finite number of max & min.



4)  $f(t)$  in a finite time interval, has only a finite number of discontinuities



## \* properties for trigonometric FS

(i) even function -

\*  $x(t) = x(-t)$ ,  $f(t)$  is even function

$$- x(t) = x(-t)$$

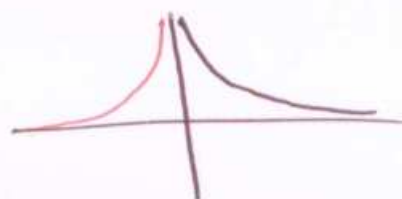
- symmetrical about y-axis

- so if the periodic signal is even function then

the trigonometric FS coefficient

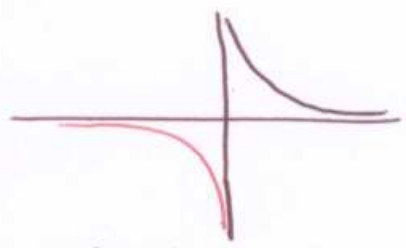
$$\boxed{b_n = 0}$$

because sin is odd



## ② Odd function:-

- $x(t) = -x(-t)$
- symmetrical about origin
- so if the periodic signal is odd function then



the trigonometric FS coefficient  $a_n$  & DC component  $a_0$  are both equal zero

$$a_0 = a_n = 0$$

ملحوظة: إذا كان الإشارة دالة فردية فإنها لا تحتوي على مكون ثابت ولا مكون جيبية.

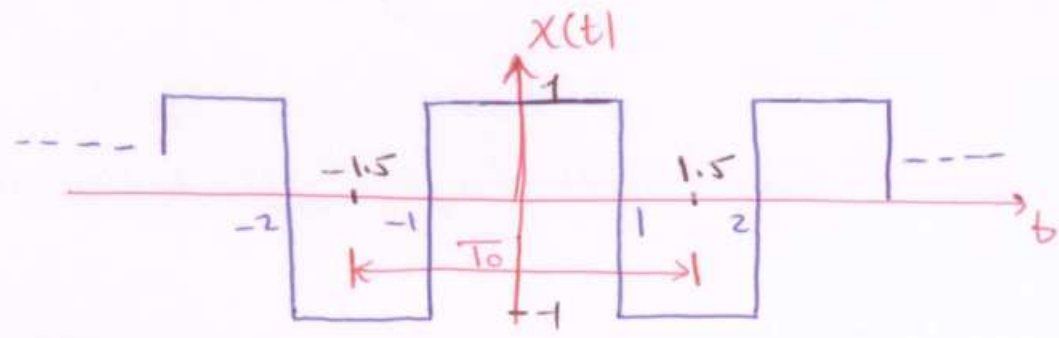
### \* Remark 2

symmetrical about line or axis

- Integrate over  $\left(\frac{T_0}{2}\right)$  only and multiply it by 2

### Example 1

Find the Fourier series



→ steps

- 1) Find the fundamental period  $T_0$

$$T_0 = 3 \text{ sec}, \text{ so the } \omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{3} \text{ rad/sec}$$

- 2) find  $x(t)$  value over one period

$$x(t) = \begin{cases} -1, & -1.5 < t < -1 \\ 1, & -1 < t < 1 \\ -1, & 1 < t < 1.5 \end{cases} \quad \left. \vphantom{\begin{cases} -1, & -1.5 < t < -1 \\ 1, & -1 < t < 1 \\ -1, & 1 < t < 1.5 \end{cases}} \right\} \text{ complete one period.}$$

3) Find the trigonometric FS coefficients-

[10]

- This signal is even function because  $x(1) = x(-1) = 1$   
 So  $a_0, a_n$  exist and  $b_n$  equal zero.

$$a_0 = \frac{1}{T_0} \int_{T_0} f(t) dt$$

$$\begin{aligned} \therefore a_0 &= \frac{1}{3} \left[ \int_{-1.5}^{-1} -1 dt + \int_{-1}^1 1 dt + \int_1^{1.5} -1 dt \right] \\ &= \frac{1}{3} \left[ -t \Big|_{-1.5}^{-1} + t \Big|_{-1}^1 + -t \Big|_1^{1.5} \right] \\ &= \frac{1}{3} \left[ (1 - 1.5) + (1 + 1) + (-1.5 + 1) \right] \end{aligned}$$

$$\boxed{a_0 = \frac{1}{3}}$$

$$a_n = \frac{2}{T_0} \int_{T_0} f(t) \cos(n\omega_0 t) dt$$

$$= \frac{2}{3} \left[ \int_{-1.5}^{-1} -\cos(n\omega_0 t) dt + \int_{-1}^1 \cos(n\omega_0 t) dt + \int_1^{1.5} -\cos(n\omega_0 t) dt \right]$$

$$= \frac{2}{3} \left[ -\frac{\sin(n\omega_0 t)}{n\omega_0} \Big|_{-1.5}^{-1} + \frac{\sin(n\omega_0 t)}{n\omega_0} \Big|_{-1}^1 + -\frac{\sin(n\omega_0 t)}{n\omega_0} \Big|_1^{1.5} \right]$$

$$\omega_0 = \frac{2\pi}{3}$$

$$\begin{aligned} &= \frac{2}{3} \left[ -\frac{\sin(-\frac{2\pi}{3}n)}{\frac{2\pi}{3}n} + \frac{\sin(-\pi n)}{\frac{2\pi}{3}n} + \frac{\sin(\frac{2\pi}{3}n)}{\frac{2\pi}{3}n} - \frac{\sin(-\frac{2\pi}{3}n)}{\frac{2\pi}{3}n} \right. \\ &\quad \left. - \frac{\sin(\pi n)}{\frac{2\pi}{3}n} + \frac{\sin(\frac{2\pi}{3}n)}{\frac{2\pi}{3}n} \right] \end{aligned}$$

$$\sin \pi n = 0$$

$$\sin -\frac{2\pi}{3}n = -\sin \frac{2\pi}{3}n$$

$$a_n = \frac{2}{3} \left[ \frac{\sin(\frac{2\pi}{3}n)}{\frac{2\pi}{3}n} + \frac{\sin(\frac{2\pi}{3}n)}{\frac{2\pi}{3}n} + \frac{\sin(\frac{2\pi}{3}n)}{\frac{2\pi}{3}n} + \frac{\sin(\frac{2\pi}{3}n)}{\frac{2\pi}{3}n} \right] \quad |||$$

$$a_n = \frac{2}{3} \left[ \frac{4 \sin(\frac{2\pi}{3}n)}{\frac{2\pi}{3}n} \right]$$

$$a_n = \frac{4 \sin(\frac{2\pi}{3}n)}{\pi n}$$

Now, we know that  $b_n = 0$  but we will verify that only one time.

$$b_n = \frac{2}{3} \int_{T_0} f(t) \sin(\omega_n t) dt$$

$$= \frac{2}{3} \left[ \int_{-1.5}^{-1} -\sin(\frac{2\pi}{3}nt) dt + \int_{-1}^1 \sin(\frac{2\pi}{3}nt) dt - \int_{1}^{1.5} \sin(\frac{2\pi}{3}nt) dt \right]$$

$$= \frac{2}{3} \left[ \frac{\cos \frac{2\pi}{3}nt}{\frac{2\pi}{3}n} \Big|_{-1.5}^{-1} - \frac{\cos \frac{2\pi}{3}nt}{\frac{2\pi}{3}n} \Big|_{-1}^1 + \frac{\cos \frac{2\pi}{3}nt}{\frac{2\pi}{3}n} \Big|_1^{1.5} \right]$$

$$= \frac{2}{3} \left[ \frac{\cos(-\frac{2\pi}{3}n)}{\frac{2\pi}{3}n} - \frac{\cos(-\pi n)}{\frac{2\pi}{3}n} - \frac{\cos \frac{2\pi}{3}n}{\frac{2\pi}{3}n} + \frac{\cos(-\frac{2\pi}{3}n)}{\frac{2\pi}{3}n} + \frac{\cos(\frac{2\pi}{3}n)}{\frac{2\pi}{3}n} - \frac{\cos \frac{2\pi}{3}n}{\frac{2\pi}{3}n} \right]$$

$$= \frac{2}{3} \cdot \frac{1}{\frac{2\pi}{3}n} \left[ \cancel{\cos \frac{2\pi}{3}n} - \cancel{\cos \pi n} - \cancel{\cos \frac{2\pi}{3}n} + \cancel{\cos \frac{2\pi}{3}n} + \cancel{\cos \pi n} - \cancel{\cos \frac{2\pi}{3}n} \right]$$

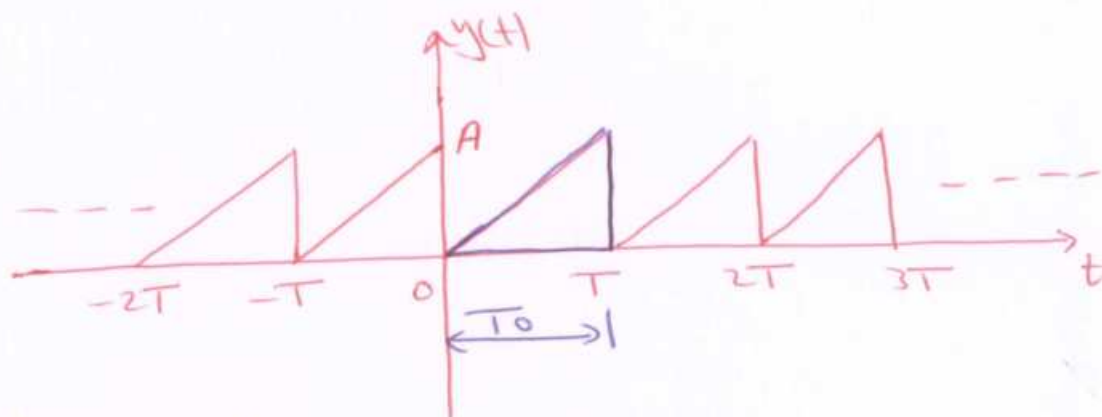
$$b_n = 0 \quad \underline{\text{yes}} \quad (\underline{\text{true}})$$

$$\therefore X(t) = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{4 \sin(\frac{2\pi}{3}n)}{\pi n} \cdot \cos \frac{2\pi n}{3} t \quad \#.$$

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### Example 22

Find the Fourier series for  $y(t)$ ?



- Slope =  $\frac{A-0}{T-0} = \frac{A}{T}$

- Fundamental period  $T_0 = T$  &  $\omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{T}$

- $y(t) = \frac{A}{T} t$  for  $0 < t < T$  (over one period)

- $y(t) = a_0 + \sum_{n=1}^{\infty} [a_n \cos \omega_0 n t + b_n \sin \omega_0 n t]$

-  $a_0 = \frac{1}{T_0} \int_{T_0} f(t) dt$

$$a_0 = \frac{1}{T} \int_0^T \frac{A}{T} t dt = \frac{1}{T} \left( \frac{A}{T} \frac{t^2}{2} \right) \Big|_0^T$$

$$a_0 = \frac{A}{2} \quad \#$$

-  $a_n = \frac{2}{T} \int_0^T \frac{A}{T} t \cdot \cos \omega_0 n t dt$

$$\therefore a_n = \frac{2}{T} \left[ \frac{A}{T} t \cdot \frac{\sin \omega_0 t}{\omega_0} + \frac{A \cos \omega_0 t}{T \omega_0} \right]_0^T$$

$\frac{A}{T} t$   $\cos \omega_0 t$   
 $\frac{A}{T}$   $\frac{\sin \omega_0 t}{\omega_0}$   
 $0$   $-\frac{\cos \omega_0 t}{\omega_0^2}$

$$a_n = \frac{2A}{T^2} \left[ T \frac{\sin \omega_0 T}{\omega_0} + \frac{\cos \omega_0 T}{\omega_0^2} - \frac{1}{\omega_0^2} \right]$$

$$= \frac{2A}{T^2} \left[ T \cdot \frac{\sin \frac{2\pi n}{T}}{\frac{2\pi n}{T}} + \frac{\cos \frac{2\pi n}{T}}{\frac{2\pi n}{T}} - \frac{1}{\frac{2\pi n}{T}} \right]$$

$$a_n = 0$$

$$b_n = \frac{2}{T} \int_0^T \frac{A}{T} t \cdot \sin \omega_0 t \, dt$$

$$= \frac{2}{T} \left[ -\frac{A}{T} t \cdot \frac{\cos \omega_0 t}{\omega_0} + \frac{A}{T} \frac{\sin \omega_0 t}{\omega_0} \right]_0^T$$

$$= \frac{2A}{T^2} \left[ -T \cdot \frac{\cos \frac{2\pi n}{T}}{\frac{2\pi n}{T}} + \frac{\sin \frac{2\pi n}{T}}{\frac{2\pi n}{T}} - 0 \right]$$

$$= \frac{2A}{T^2} \left[ -\frac{T^2}{2\pi n} + 0 \right] = -\frac{2A}{2\pi n}$$

$$\therefore \boxed{b_n = -\frac{A}{\pi n}}$$

$$\therefore y(t) = \frac{A}{2} + \sum_{n=1}^{\infty} \left[ -\frac{A}{\pi n} \sin \omega_0 t \right]$$

#.

## [2] Complex Exponential Fourier series

[14]

- Its another way to represent the signal in frequency domain
- Its Expression

$$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

where

$$C_n = \frac{1}{T_0} \int_{T_0} f(t) \cdot e^{-jn\omega_0 t} dt \quad \text{at } n \neq 0$$

$$C_0 = \frac{1}{T_0} \int_{T_0} f(t) dt \quad \text{at } n = 0$$

1st  $C_0$  is the average value of  $f(t)$  over one period  $T_0$ .

### Some Properties

- $C_n$  is complex number has the following formula.

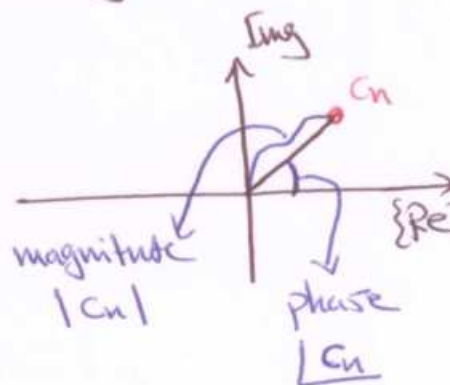
(Rectangular form)

$$C_n = \alpha_n + j\beta_n$$

- So we can represent  $C_n$  as

(Polar form)

$$C_n = |C_n| \angle C_n$$



- if  $f(t)$  is real function then there no complex component

$$C_{-n} = C_n^*$$

- if  $f(t)$  is even function  $\Rightarrow$   $C_n$  is real
- if  $f(t)$  is odd function  $\Rightarrow$   $C_n$  is Imag
- The Relation between the two ways

1)  $a_0 = C_0$  (special case & it's the average value of the signal over one period & it's equal the DC-component at  $n=0$ )

2)  $a_n = 2 \operatorname{Re}\{C_n\}$

3)  $b_n = -2 \operatorname{Im}\{C_n\}$

4)  $C_n = \frac{1}{2} [a_n - j b_n]$

Tip

Some laws or rules

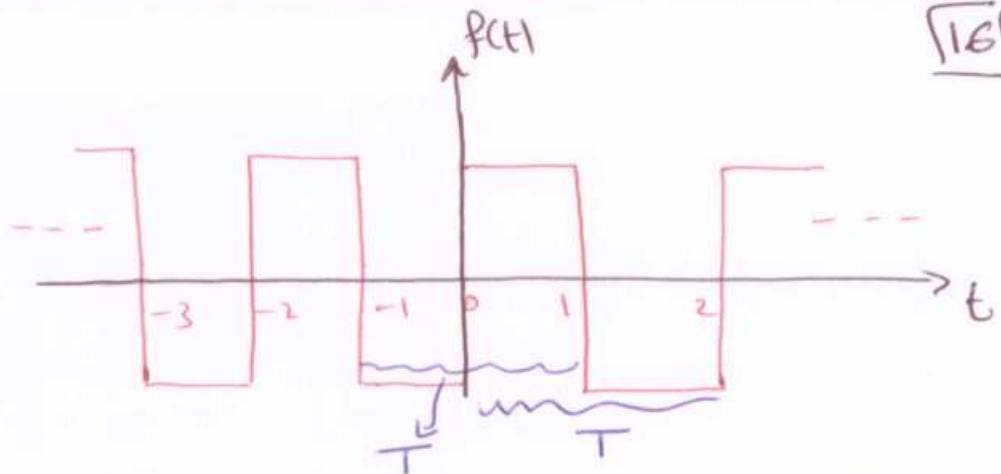
1)  $e^{j\omega t} = \cos \omega t + j \sin \omega t$

2)  $\cos \omega t = \frac{1}{2} [e^{j\omega t} + e^{-j\omega t}]$

3)  $\sin \omega t = \frac{1}{2j} [e^{j\omega t} - e^{-j\omega t}]$

### Example 3.2

Find the Fourier series for  $f(t)$  using exponential



del given  $\frac{T}{2}$   $\frac{T}{2}$  is plus 1

•  $T_0 = 2$  ,  $\omega_0 = \pi$

•  $f(t) = \begin{cases} -1 & , -1 < t < 0 \\ +1 & , 0 < t < 1 \end{cases}$  over one period.

•  $f(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\omega_0 n t}$

$\therefore c_n = \frac{1}{T} \int_{T_0} f(t) e^{-j\omega_0 n t} dt$

$c_n = \frac{1}{2} \left[ \int_{-1}^0 -e^{-j\omega_0 n t} dt + \int_0^1 +e^{-j\omega_0 n t} dt \right]$

$= \frac{1}{2} \left[ \left. \frac{-e^{-j\omega_0 n t}}{-j\omega_0 n} \right|_{-1}^0 + \left. \frac{e^{-j\omega_0 n t}}{-j\omega_0 n} \right|_0^1 \right]$

$= \frac{1}{2j\pi n} \left[ +e^{-j\pi n} + e^{j\pi n} + (-e^{-j\pi n}) + 1 \right]$

$= \frac{1}{2j\pi n} \left[ 2 - e^{j\pi n} - e^{-j\pi n} \right]$

$c_n = \frac{j}{\pi n} \left[ -1 + \frac{e^{+j\pi n} + e^{-j\pi n}}{2} \right] = \frac{j}{\pi n} [-1 + \cos \pi n]$

$$\therefore f(t) = \sum_{n=-\infty}^{\infty} \frac{j}{\pi n} [\cos \pi n - 1] \cdot e^{j\pi n t}$$

$n \Rightarrow$  integer number

The average value of the signal

$$\begin{aligned} C_0 &= \frac{1}{2} \left[ \int_{-1}^0 -1 dt + \int_0^1 1 dt \right] \\ &= \frac{1}{2} \left[ -t \Big|_{-1}^0 + t \Big|_0^1 \right] \\ &= \frac{1}{2} [0 - (-1) + 1 - 0] = 0 \end{aligned}$$

$\therefore a_0 = a_n = C_0 = \text{zero}$  because the signal is odd

$\neq C_n$  is  $\text{Imag}$

Notes:

$$C_n = \frac{j}{\pi n} [\cos \pi n - 1]$$

if  $n$  is even (2, 4, 6, ...)

1

if  $n$  is odd (+1, +3, 5, ...)

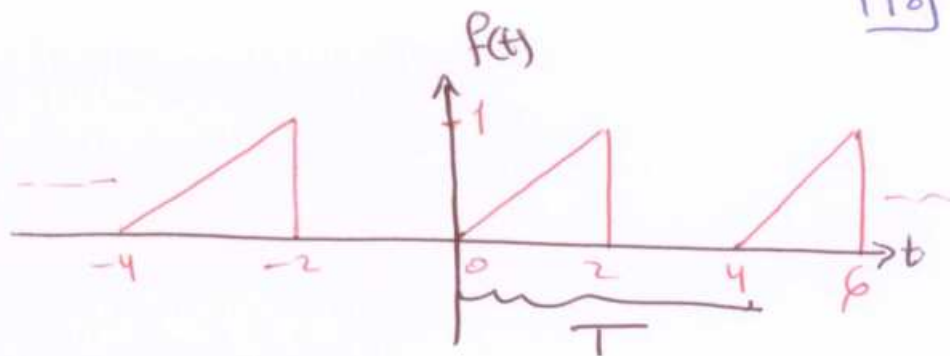
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$$\therefore C_n = \begin{cases} 0 & \text{if } n \text{ even} \\ \frac{2}{j\pi n} & \text{if } n \text{ odd} \end{cases}$$

$\#$

Example 4.2

Find the Fourier series using exponential.



$$\bullet T_0 = 4, \quad \omega_0 = \frac{2\pi}{4} = \frac{\pi}{2}$$

$$\bullet f(t) = \begin{cases} \frac{1}{2}t & , 0 < t < 2 \\ 0 & , 2 < t < 4 \end{cases}$$

$$c_n = \frac{1}{4} \left[ \int_0^2 \frac{1}{2}t e^{-j\omega_0 n t} dt \right]$$

$$c_n = \frac{1}{8} \left[ \frac{-t \cdot e^{-j\omega_0 n t}}{j\omega_0 n} + \frac{e^{-j\omega_0 n t}}{\omega_0^2 n^2} \right]_0^2$$

$$\begin{array}{l} t \quad \frac{-j\omega_0 n t}{e} \\ 1 \quad \frac{-j\omega_0 n}{e} \\ 0 \quad \frac{-j\omega_0 n}{e} \\ \quad \frac{-\omega_0^2 n^2}{e} \end{array}$$

$$c_n = \frac{1}{8} \left[ \frac{-2e^{-j\omega_0 n(2)}}{j\omega_0 n} + \frac{e^{-j\omega_0 n(2)}}{\omega_0^2 n^2} - 0 - \frac{1}{\omega_0^2 n^2} \right]$$

$$= \frac{1}{8} \left[ -\frac{2e^{-j\pi n}}{j\frac{\pi}{2}n} + \frac{e^{-j\pi n}}{\frac{\pi^2}{4}n^2} - \frac{1}{\frac{\pi^2}{4}n^2} \right] \quad \text{pls } \frac{\pi^2 n^2}{4} \text{ is } \frac{1}{\frac{4}{\pi^2 n^2}}$$

$$= \frac{1}{28} \cdot \frac{1}{\frac{\pi^2}{4}n^2} \left[ +2 \cdot j \cdot \frac{\pi}{2}n \cdot e^{-j\pi n} + e^{-j\pi n} - 1 \right]$$

$$e^{-j\pi n} = \cos(-\pi n) + j \sin(-\pi n)$$

$$= \cos \pi n - j \sin \pi n$$

For  $\forall n$  even or odd

$$\therefore C_n = \frac{1}{2\pi^2 n^2} \left[ j\pi n \cdot e^{-j\pi n} + e^{-j\pi n} - 1 \right]$$

(19)

$$C_n = \frac{1}{2\pi^2 n^2} \left[ j\pi n \cdot \cos \pi n + \cos \pi n - 1 \right] \text{ at } n \neq 0$$

at  $n=0$

$C_0 = \frac{1}{0}$  undefined so we will calculate by the rule

$$C_0 = \frac{1}{4} \int_0^2 \frac{1}{2} t \, dt = \frac{1}{4} \cdot \frac{1}{2} \cdot \frac{t^2}{2} \Big|_0^2 = \frac{1}{16} (4 - 0) = \left[ \frac{1}{4} \right]$$

$$\therefore C_n = \begin{cases} \frac{1}{4}, & n=0 \\ \frac{j}{2\pi n}, & n\text{-even} \\ \frac{-1}{2\pi^2 n^2} [j\pi n + 2], & n\text{-odd} \end{cases}$$

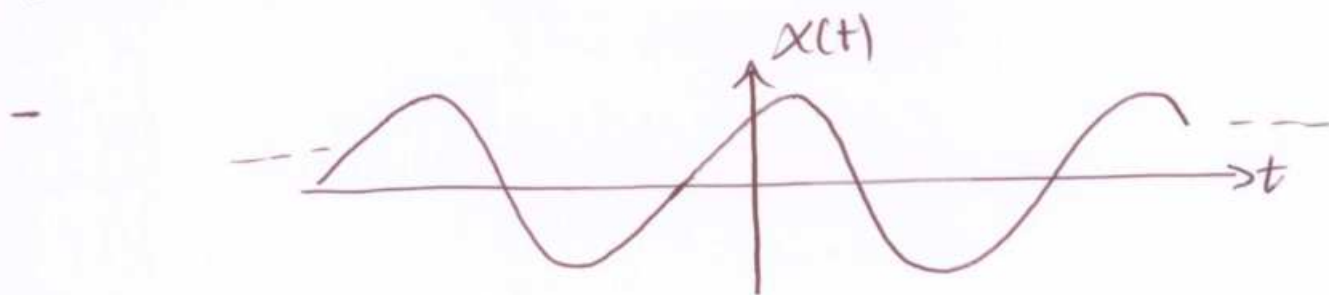
$$\cos \pi n = 1 \quad \text{even} \\ = -1 \quad \text{odd}$$

$$\cos \pi n = -1 \quad \text{odd}$$

$$\therefore f(t) = \begin{cases} \frac{1}{4} + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{j}{2\pi n} e^{j\frac{\pi}{2}nt} & \underline{n\text{-even}} \\ \frac{1}{4} + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{-1}{2n^2\pi^2} [j\pi n + 2] \cdot e^{j\frac{\pi}{2}nt} & \underline{n\text{-odd}} \end{cases}$$

*Handwritten signature*

⊛ line spectra : (Amplitude spectrum & magnitude spectrum) 20  
spectrum is  $\omega$   $\leftarrow$



$$x(t) = A \sin(\omega t + \phi)$$

$$\begin{aligned} &\Updownarrow \\ x(\omega) &= A \angle \phi \Rightarrow \begin{array}{l} \text{Amplitude Spectrum} \\ \text{Phase Spectrum} \end{array} \end{aligned}$$

- it is used to sketch the spectrum around frequency & it helps us in design and compute the Bandwidth.
- The complex exponential Fourier series of a periodic signal consists of a summation of phasors (Mag/phase)
- The periodic signal to be characterized graphically in the frequency domain is by making 2 plots
  - 1) First, showing Amplitudes, versus frequency (Amplitude spectrum)
  - 2) Second, showing phase, versus frequency (phase spectrum)

Notes

- Amplitude spectrum  $|C_n| = |C_{-n}|$  is symmetrical about y-axis  
 $\uparrow$  even function
- phase spectrum  $\angle C_n = -\angle C_{-n}$  is symmetrical about origin  
 $\uparrow$  odd function

Ex 1 Back To the previous example & sketch the line spectra.

$$C_n = \begin{cases} \frac{1}{4} & , n=0 \\ \frac{j}{2\pi n} & , n\text{-even} \\ \frac{-1}{2\pi^2 n^2} [j\pi n + 2] & , n\text{-odd} \end{cases}$$

$n = 0, \pm 1, \pm 2, \pm 3, \pm 4, \dots$

$$C_0 = \frac{1}{4} \angle 0$$

$$C_1 = 0.189 \angle -122.5$$

$$C_2 = 0.08 \angle 90$$

$$C_3 = 0.054 \angle -102$$

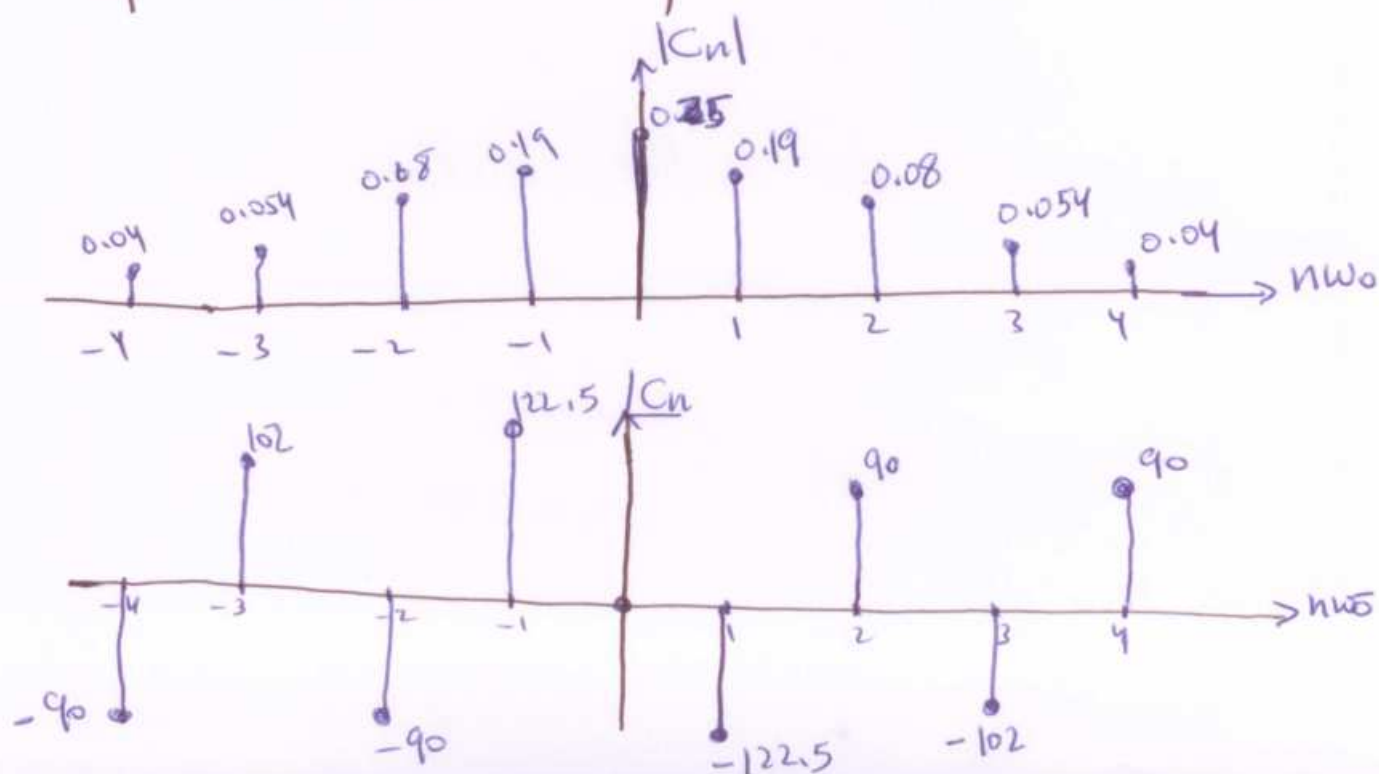
$$C_4 = 0.04 \angle 90$$

$$C_{-1} = 0.189 \angle 122.5$$

$$C_{-2} = 0.08 \angle -90$$

$$C_{-3} = 0.054 \angle 102$$

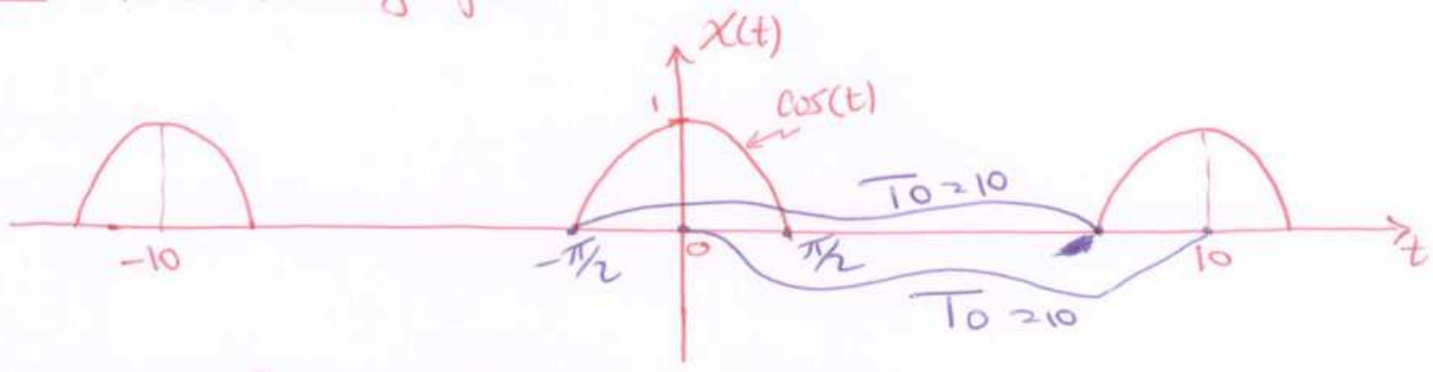
$$C_{-4} = 0.04 \angle -90$$



- ②  $\omega, p$  سوال این است
- ① است  $f(t)$  را به  $\omega$  و  $p$  تبدیل کنید
- ③ است  $C_n$  را به  $\omega$  و  $p$  تبدیل کنید
- ④  $\omega$  و  $p$  را به  $\omega$  و  $p$  تبدیل کنید

Ex 2

For the following signal:



- a) Find the Fourier series
- b) plot the spectra versus frequency,  $\omega = \omega_0 n$

•  $T_0 = 10$  ,  $\omega_0 = \frac{\pi}{5}$

•  $x(t) = \begin{cases} \cos(t) & -\frac{\pi}{2} < t < \frac{\pi}{2} \\ 0 & \text{elsewhere} \end{cases}$

•  $x(t) = \sum_{n=-\infty}^{\infty} C_n e^{j\omega_0 n t}$

$C_n = \frac{1}{10} \left[ \int_{-\pi/2}^{\pi/2} \cos(t) \cdot e^{-j\omega_0 n t} dt \right]$

$C_n = \frac{1}{10} \left[ \int_{-\pi/2}^{\pi/2} \frac{1}{2} [e^{jt} + e^{-jt}] e^{-j\omega_0 n t} dt \right]$

$= \frac{1}{20} \left[ \int_{-\pi/2}^{\pi/2} e^{j(t - \omega_0 n t)} dt + \int_{-\pi/2}^{\pi/2} e^{-j(t + \omega_0 n t)} dt \right]$

(From the table of integration)

$\cos(t) e^{-j\omega_0 n t}$

فرد است و زوج نیست

Properties

$$C_n = \frac{1}{20} \left[ \int_{-\pi/2}^{\pi/2} e^{j(t - \omega_0 t)} dt + \int_{-\pi/2}^{\pi/2} e^{-j(t + \omega_0 t)} dt \right]$$

$$= \frac{1}{20} \left[ \int_{-\pi/2}^{\pi/2} e^{jt(1 - \omega_0)} dt + \int_{-\pi/2}^{\pi/2} e^{-jt(1 + \omega_0)} dt \right]$$

$$= \frac{1}{20} \left[ \frac{e^{jt(1 - \omega_0)}}{j(1 - \omega_0)} \Big|_{-\pi/2}^{\pi/2} + \frac{e^{-jt(1 + \omega_0)}}{-j(1 + \omega_0)} \Big|_{-\pi/2}^{\pi/2} \right]$$

$$= \frac{1}{20} \cdot \frac{1}{j(1 - \omega_0)} \left[ \frac{j(\pi/2 - \pi/2 \omega_0)}{e} - \frac{-j(\pi/2 - \pi/2 \omega_0)}{e} \right]$$

note  
 $(e^{jx} = \cos x + j \sin x)$

$$= \frac{1}{20j(1 - \omega_0)} \left[ \frac{-j\pi/2(1 + \omega_0)}{e} - \frac{+j\pi/2(1 + \omega_0)}{e} \right]$$

$$= \frac{1}{20j(1 - \omega_0)} \left[ \cancel{e^{+j\pi/2}} \cdot e^{-j\pi/2 \omega_0} - \cancel{e^{-j\pi/2}} \cdot e^{+j\pi/2 \omega_0} \right]$$

$$= \frac{1}{20j(1 - \omega_0)} \left[ \cancel{e^{-j\pi/2}} \cdot e^{-j\pi/2 \omega_0} - \cancel{e^{+j\pi/2}} \cdot e^{+j\pi/2 \omega_0} \right]$$

$$= \frac{1}{20j(1 - \omega_0)} \left( \cancel{j} e^{-j\pi/2 \omega_0} + \cancel{j} e^{j\pi/2 \omega_0} \right) + \left( \frac{\cancel{j} e^{-j\pi/2 \omega_0} + \cancel{j} e^{j\pi/2 \omega_0}}{20j(1 + \omega_0)} \right)$$

$$= \frac{1}{10(1 - \omega_0)} \left( \cancel{2} \cos\left(\frac{\pi}{2} \omega_0\right) \right) + \left( \frac{\cancel{2} \cos\left(\frac{\pi}{2} \omega_0\right)}{10(1 + \omega_0)} \right)$$

$$\therefore C_n = \frac{\cos(\pi/2 \omega_0)}{10(1 - \omega_0)} + \frac{\cos(\pi/2 \omega_0)}{10(1 + \omega_0)}$$

$$\begin{aligned}
 C_n &= \cos\left(\frac{\pi}{2} \omega_0 n\right) \left[ \frac{1}{10(1-\omega_0 n)} + \frac{1}{10(1+\omega_0 n)} \right] \\
 &= \cos\left(\frac{\pi}{2} \omega_0 n\right) \left[ \frac{10(1+\omega_0 n) + 10(1-\omega_0 n)}{100((1-\omega_0 n)(1+\omega_0 n))} \right] \\
 &= \cos\left(\frac{\pi}{2} \omega_0 n\right) \left[ \frac{20}{500(1-(\omega_0 n)^2)} \right]
 \end{aligned}$$

$$C_n = \frac{\cos \frac{\pi}{2} \omega_0 n}{5(1-(\omega_0 n)^2)} = \frac{\cos\left(\frac{\pi^2}{10} n\right)}{5(1-(\frac{n\pi}{5})^2)}$$

$$a) \quad x(t) = \frac{1}{5} + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{\cos\left(\frac{\pi^2}{10} n\right)}{5(1-(\frac{n\pi}{5})^2)} \cdot e^{j\frac{\pi}{5} n t}$$

$$b) \quad C_0 = \frac{1}{5} L_0$$

$$C_1 = 0.182 L_0$$

$$C_2 = 0.135 L_0$$

$$C_3 = 0.07 L_0$$

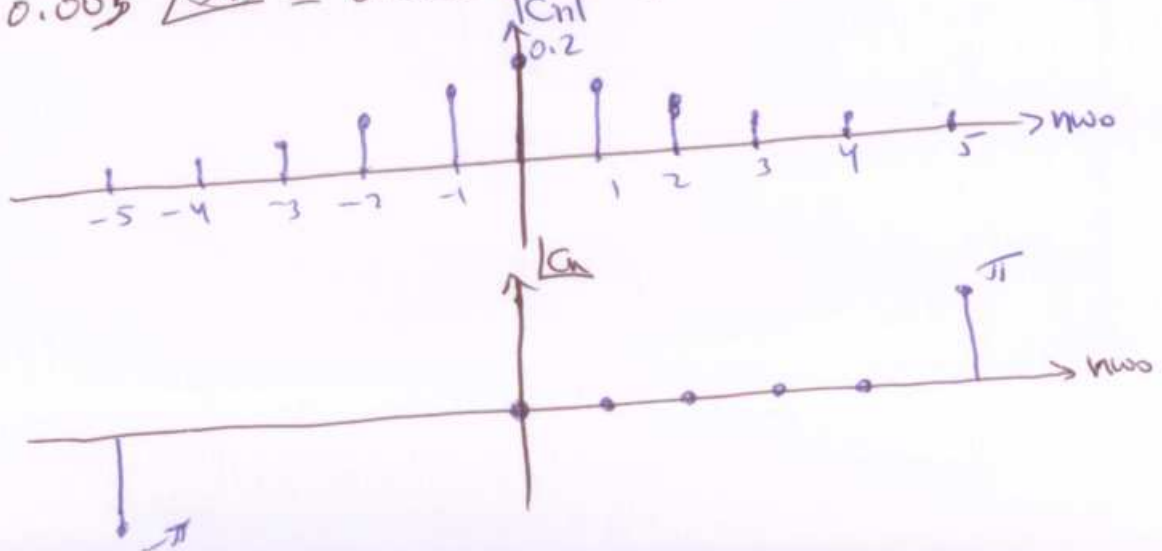
$$C_4 = 0.026 L_0$$

$$C_5 = 0.005 \sqrt{3.14} = 0.005 \frac{\sqrt{\pi}}{L_0} \text{ radian}, \quad C_5 = 0.005 L_{-\pi}$$

No Angle No Imag

لا يوجد زاوية ولا تخيل

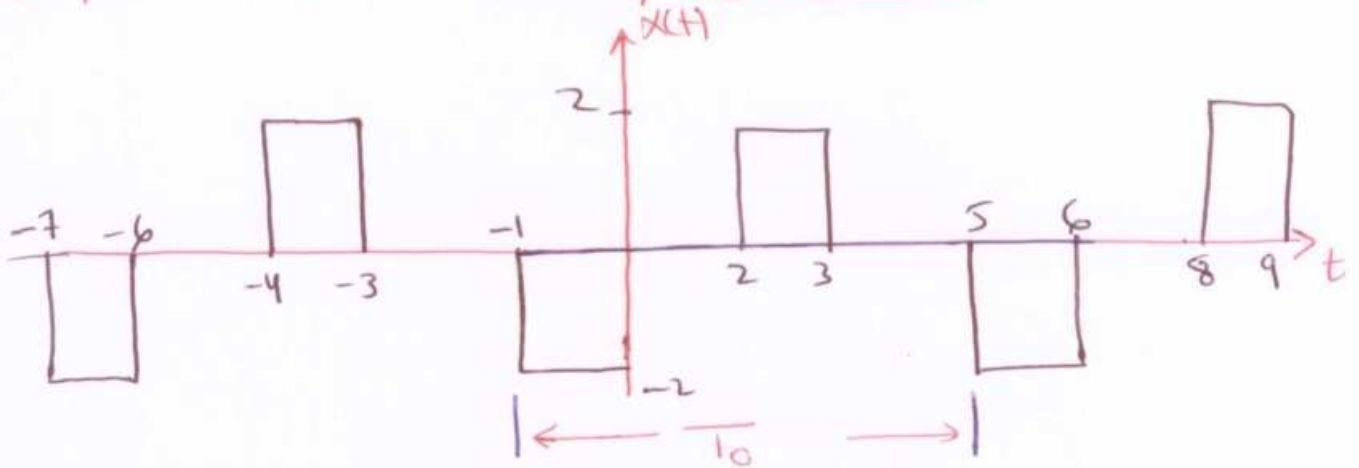
الزاوية من راديان  
لا يوجد تخيل



# Ex 2 # For the following signals

a) Find the Fourier series (Trigonometric)

b) plot the spectra versus frequency  $\omega = n\omega_0$



$$\omega_0 = \frac{2\pi}{T_0}, \quad T_0 = \boxed{6 \text{ sec}} \Rightarrow \omega_0 = \frac{2\pi}{6} = \boxed{\frac{\pi}{3}}$$

a)

$$x(t) = \begin{cases} -2 & -1 \leq t < 0 \\ 0 & 0 \leq t < 2 \\ 2 & 2 \leq t < 3 \\ 0 & 3 \leq t < 5 \end{cases}$$

$$\begin{aligned} a_0 &= \frac{1}{6} \int_{-1}^5 x(t) dt = \frac{1}{6} \left[ \int_{-1}^0 -2 dt + \int_2^3 2 dt \right] \\ &= \frac{1}{6} \left[ -2t \Big|_{-1}^0 + 2t \Big|_2^3 \right] = \frac{1}{6} (-2 + 2) = \frac{0}{6} = \boxed{0} \end{aligned}$$

$$a_n = \frac{2}{T} \int_{T_0} x(t) \cos \omega_0 n t \, dt$$

$$a_n = \frac{2}{6} \left[ \int_{-1}^0 -2 \cos \omega_0 n t \, dt + \int_2^3 2 \cos \omega_0 n t \, dt \right]$$

$$= \frac{1}{3} \left[ -\frac{2 \sin \omega_0 n t}{\omega_0 n} \Big|_{-1}^0 + \frac{2 \sin \omega_0 n t}{\omega_0 n} \Big|_2^3 \right]$$

$$= \frac{1}{3} \left[ 0 - \frac{2 \sin \omega_0 n}{\omega_0 n} + 2 \frac{\sin \omega_0 n (3)}{\omega_0 n} - \frac{2 \sin 2 \omega_0 n}{\omega_0 n} \right]$$

$$a_n = \frac{2}{3 \cdot \frac{\pi}{3} n} \left[ \cancel{\sin 3 \frac{\pi}{3} n} - \sin 2 \cdot \frac{\pi}{3} n - \sin \frac{\pi}{3} n \right]$$

$$a_n = \frac{2}{\pi n} \left[ -\sin \frac{2\pi}{3} n - \sin \frac{\pi}{3} n \right] \#$$

$$b_n = \frac{2}{T} \int_{T_0} x(t) \sin \omega_0 n t \, dt$$

$$= \frac{2}{6} \left[ \int_{-1}^0 -2 \sin \omega_0 n t \, dt + \int_2^3 2 \sin \omega_0 n t \, dt \right]$$

$$= \frac{1}{3} \left[ \frac{2 \cos \omega_0 n t}{\omega_0 n} \Big|_{-1}^0 + -\frac{2 \cos \omega_0 n t}{\omega_0 n} \Big|_2^3 \right]$$

$$= \frac{1}{3} \left[ \frac{2}{\omega_0 n} - \frac{2 \cos \omega_0 n}{\omega_0 n} - \frac{2 \cos 3 \omega_0 n}{\omega_0 n} + \frac{2 \cos 2 \omega_0 n}{\omega_0 n} \right]$$

$$b_n = \frac{2}{3 \cdot \frac{\pi}{3} n} \left[ 1 - \cos \frac{\pi}{3} n - \cos \pi n + \cos \frac{2\pi}{3} n \right] \quad (27)$$

$$b_n = \frac{2}{\pi n} \left[ 1 - \cos \frac{\pi}{3} n - \cos \pi n + \cos \frac{2\pi}{3} n \right]$$

$$C_n = \frac{1}{2} (a_n - j b_n)$$

$$\therefore C_n = \frac{1}{2} \left[ \frac{2}{\pi n} \left( -\sin \frac{2\pi}{3} n - \sin \frac{\pi}{3} n \right) - j \left( 1 - \cos \frac{\pi}{3} n - \cos \pi n + \cos \frac{2\pi}{3} n \right) \right]$$

$$C_0 = 0 = a_0$$

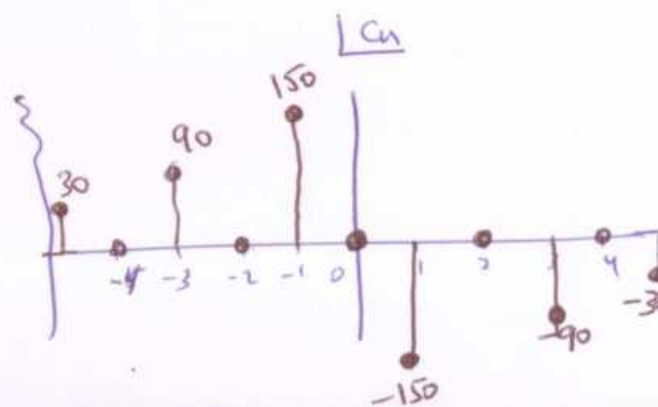
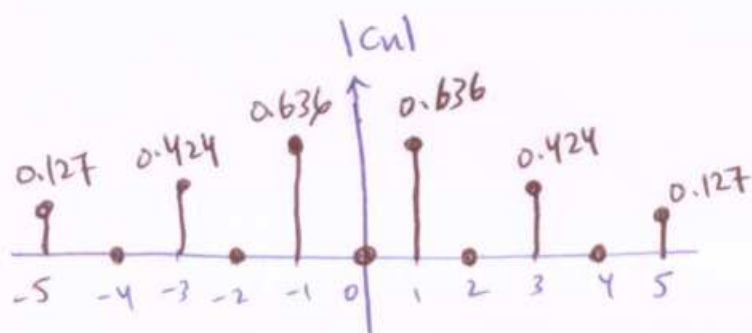
$$C_1 = \frac{1}{\pi} [-1.732 - j] = 0.636 \angle -150$$

$$C_2 = \frac{1}{2\pi} [0 - j0] = 0 \angle 0$$

$$C_3 = \frac{1}{3\pi} [0 - j4] = 0.424 \angle -90$$

$$C_4 = 0$$

$$C_5 = \frac{1}{5\pi} [1.732 - j1] = 0.127 \angle -30$$



## \* Some Important notes

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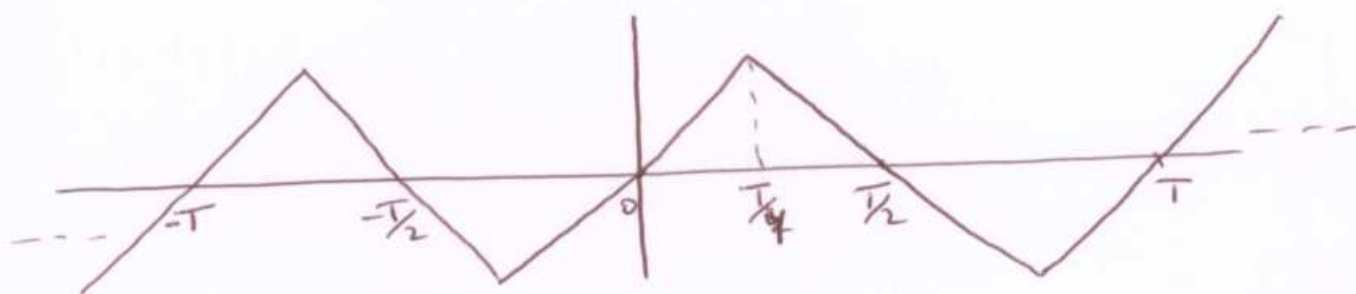
- Half-wave odd symmetry

$$x(t) = -x(t \pm T/2)$$

$$a_0 = 0, \quad a_{2n} = 0, \quad a_{2n+1} \neq 0, \quad b_{2n} = 0, \quad b_{2n+1} \neq 0$$

like the previous example.

e.g



This signal called half wave odd

### Remark 2

In even & odd & half wave odd function you can integrate over  $T/2$  only then multiply it by 2

- Compact trigonometric Fourier series

$$f(t) = D_0 + \sum_{n=1}^{\infty} D_n \cos(n\omega_0 t + \theta_n)$$

where

$$D_0 = a_0$$

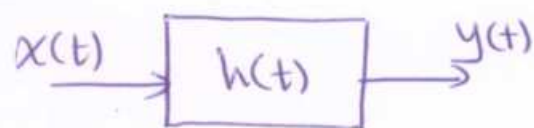
$$D_n = \sqrt{a_n^2 + b_n^2}$$

$$\theta_n = \tan^{-1}\left(\frac{b_n}{a_n}\right)$$

# \* LTI Systems Response To periodic Input

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$$y(t) = \int_{-\infty}^{\infty} x(\tau) h(t - \tau) d\tau$$



- By using convolution method to get the output signal.
- if  $x(t)$  is complex exponential  $\Rightarrow x(t) = e^{j\omega t}$

So

$$y(t) = \int_{-\infty}^{\infty} h(\tau) e^{j\omega(t-\tau)} d\tau$$

$$y(t) = e^{j\omega t} \int_{-\infty}^{\infty} h(\tau) e^{-j\omega\tau} d\tau$$

Let  $H(\omega) = \int_{-\infty}^{\infty} h(\tau) e^{-j\omega\tau} d\tau$

$$\boxed{y(t) = H(\omega) e^{j\omega t}}$$

Where  $H(\omega)$  called the system Transfer Function (T.F) and it is constant for fixed  $\omega$

- So the output  $y(t)$  for periodic input  $x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$

$$\boxed{y(t) = \sum_{n=-\infty}^{\infty} \underline{H(n\omega_0)} c_n e^{jn\omega_0 t}}$$

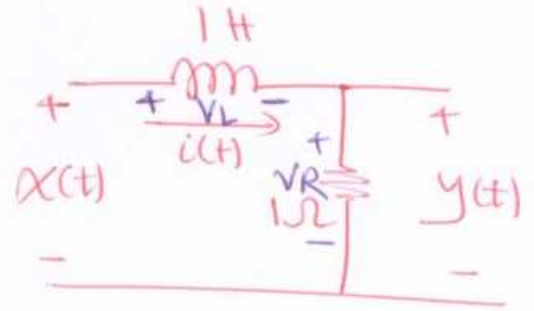
input  $\rightarrow c_n e^{jn\omega_0 t}$   
 $\rightarrow H(\omega) \rightarrow y$   
T.F  $\rightarrow \omega$

### Example

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Given  $x(t) = 4\cos t - 2\cos 2t$

Find  $y(t)$



$$x(t) - L \frac{di}{dt} - y(t) = 0$$

$$y(t) = R i(t) \Rightarrow \frac{dy}{dt} = R \frac{di}{dt} \Rightarrow \frac{di}{dt} = \frac{1}{R} \frac{dy}{dt}$$

$$x(t) - \frac{L}{R} \frac{dy}{dt} - y(t) = 0$$

$$\frac{dy}{dt} + \frac{R}{L} y(t) = \frac{R}{L} x(t) \quad \text{---} \textcircled{*}$$

Now put  $x(t) = e^{j\omega t}$  to find  $H(\omega)$

$$\therefore y(t) = H(\omega) e^{j\omega t}$$

$$\frac{d}{dt} [H(\omega) e^{j\omega t}] + \frac{R}{L} [H(\omega) e^{j\omega t}] = \frac{R}{L} e^{j\omega t}$$

$$H(\omega) \cancel{e^{j\omega t}} \cdot j\omega + \frac{R}{L} H(\omega) \cancel{e^{j\omega t}} = \frac{R}{L} \cancel{e^{j\omega t}}$$

$$H(\omega) \left( \cancel{j\omega} + \frac{R}{L} \right) = \frac{R}{L}$$

$$\boxed{H(\omega) = \frac{R/L}{(R/L + j\omega)}}$$

$$\therefore H(n\omega_0) = \frac{R/L}{(R/L + jn\omega_0)}, \quad \text{Assume } \boxed{\frac{R}{L} = 1}, \boxed{\omega_0 = 1}$$

From the circuit

$$x(t) = 4 \cos t - 2 \cos 2t$$

$$T_1 = \frac{2\pi}{1} = 2\pi$$

$$T_2 = \frac{2\pi}{2} = \pi$$

$$\frac{T_1}{T_2} = \frac{2\pi}{\pi} = \frac{2}{1} \quad \text{rational}$$

$$T_0 = 2T_2 = T_1 = 2\pi$$

$$\boxed{\omega_0 = \frac{2\pi}{T_0} = 1}$$

fundamental radian frequency

$$\therefore H(n\omega_0) = \frac{1}{1+jn}$$

$$\therefore y(t) = H(n\omega_0) e^{jn\omega_0 t}$$

$$\text{where } x(t) = 4 \cos t - 2 \cos 2t = 4 \cos t - 2 \cos 2t$$

$$\therefore y(t) = \frac{4}{1+j} \cos t - \frac{2}{1+j2} \cos 2t$$

First harmonic  $(n=1)$       Second harmonic  $(n=2)$

Phasor Form

$$y(t) = 2.83 \angle -45^\circ - 0.89 \angle -63.4^\circ$$

Rectangular Form

$$y(t) = 2.83 \cos(t - 45^\circ) - 0.89 \cos(2t - 63.4^\circ)$$

1st      2nd

أيضا،  $H(n\omega_0)$  هي دالة النقل  $n$  من المدخل إلى المخرج



Good luck