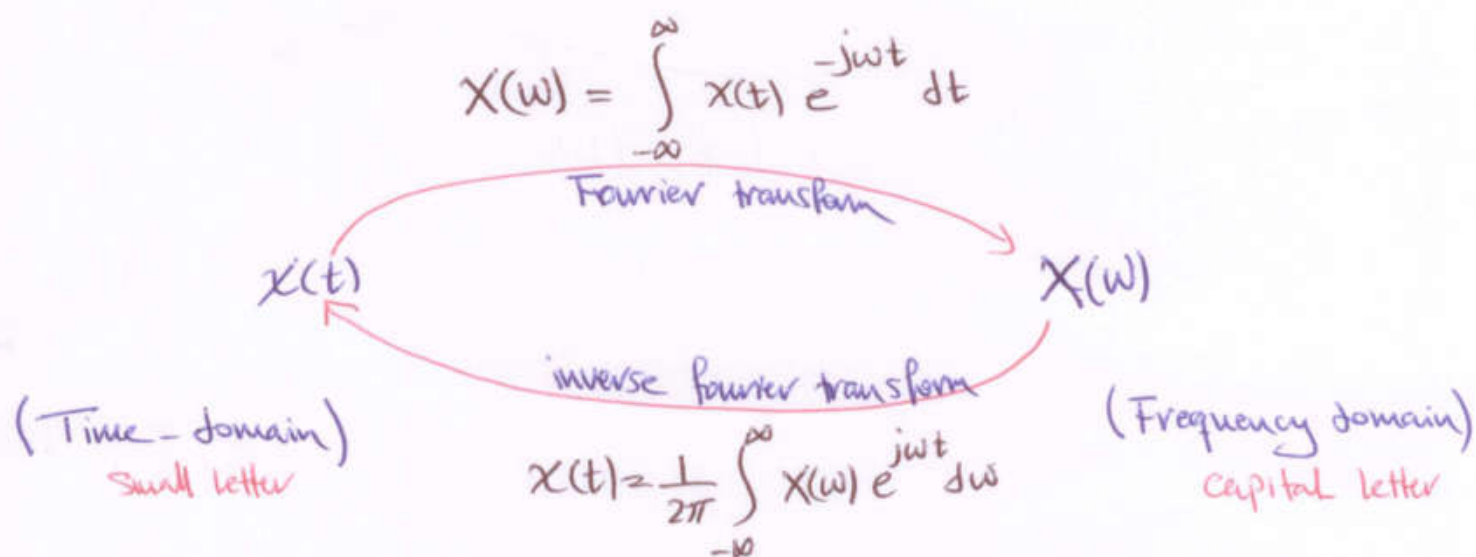


Fourier Transform

→ As, we see in chapter 6 the Fourier analysis its a type of frequency representation for signal and have two types:

- 1) Fourier series (FS) ⇒ For Periodic signal ch. 6
- 2) Fourier transform (FT) ⇒ For Aperiodic signal ch. 7

→ So, we can use Fourier analysis to move from time domain to frequency domain or to move back.



→ Fourier series enable us to describe the periodic function in terms of the frequency domain, & Fourier transform extends this frequency domain description to function that are not periodic.

• Definitions

[2]

$$\underbrace{X(\omega)}_{\text{F. domain}} = \mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

التحويل إلى التردد (t)

$$\underbrace{x(t)}_{\text{T. domain}} = \mathcal{F}^{-1}\{X(\omega)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

التحويل إلى الزمن (ω)

$X(\omega) \Rightarrow$ Fourier transform

$x(t) \Rightarrow$ inverse fourier transform

هذه التحويلات تسمى
التحويل إلى التردد و التحويل إلى الزمن
العكس والعكس

→ The function $x(t)$ has fourier transform if the integral converges

أيضا هذا الشرط
هو الشرط الجلي

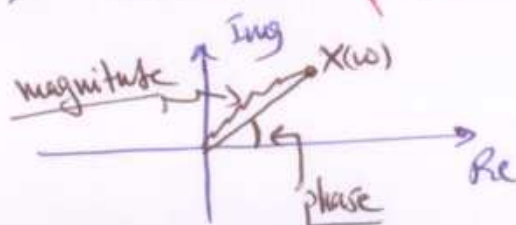
$$\int_{-\infty}^{\infty} |x(t)| dt < \infty$$

→ إذا كان
diverges

→ $X(\omega)$ is generally a complex value function of ω
that mean it has two parts (real part , Img part)

- magnitude spectrum $\Rightarrow |X(\omega)|$

- phase spectrum $\Rightarrow \angle X(\omega)$ (its in radian between $(-\pi, \pi)$)



→ The Fourier transform is a special case of the Laplace transform (LT)

So if I know the Laplace transform for $x(t)$

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

So to obtain Fourier transform replace s by $j\omega$
 $\boxed{s = j\omega}$

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

So if I know $X(s)$, I can find $X(\omega)$ very easily

Ex 2

$$x(t) = e^{-at} u(t)$$

From table

$$X(s) = \frac{1}{s+a}$$

this is the Laplace transform for $x(t)$

To find Fourier transform replace $s = j\omega$

$$\boxed{X(\omega) = \frac{1}{j\omega + a}}$$

this is the Fourier transform for $x(t)$

⊛ Transform of some useful functions:

II) a) Fourier transform of unit impulse $\delta(t)$

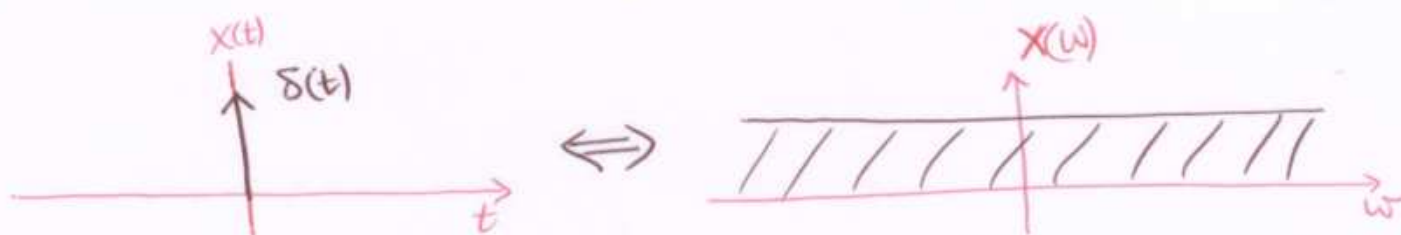
By using sampling property or shifting property

$$\int_{t_0-\epsilon}^{t_0+\epsilon} \delta(t-t_0) X(t) dt = X(t_0)$$

$$\therefore \mathcal{F}\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t) \cdot e^{-j\omega t} dt = e^{-j\omega \times 0} = 1$$

$t_0=0$

$$\therefore \underbrace{\delta(t)}_{\text{T.D}} \iff \underbrace{1}_{\text{F.D}}$$

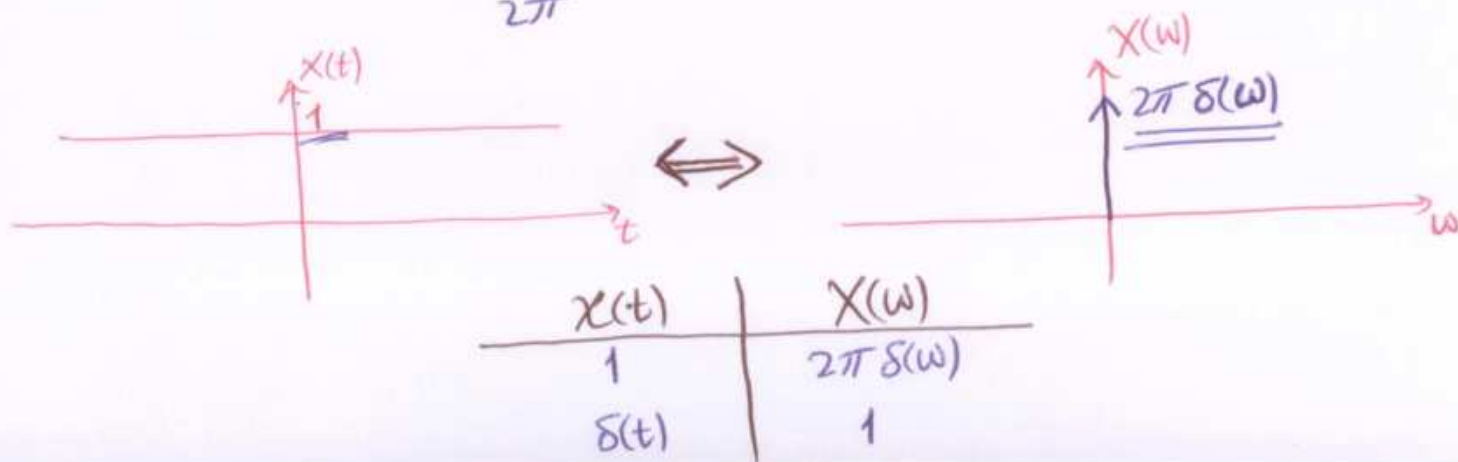


b) inverse Fourier transform of $\delta(w)$

$$\mathcal{F}^{-1}\{\delta(w)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(w) e^{+j\omega t} dw = \frac{1}{2\pi}$$

$w_0=0$

$$\therefore \frac{1}{2\pi} \iff \delta(w) \text{ OR } 1 \iff 2\pi \delta(w)$$



2 Fourier transform of $x(t) = \text{rect}(\frac{t}{T})$

$t_0 = 0 \rightarrow$ center at $t = 0$ & length T (period)
 & $A = 1$

$$X(\omega) = \int_{-\infty}^{\infty} \text{rect}(\frac{t}{T}) e^{-j\omega t} dt$$

$$= \int_{-T/2}^{T/2} 1 \cdot e^{-j\omega t} dt = \frac{e^{-j\omega t}}{-j\omega} \Big|_{-T/2}^{T/2}$$

$$= \frac{1}{-j\omega} \left[e^{-j\omega \cdot T/2} - e^{+j\omega \cdot T/2} \right] \times \frac{-1}{-1}$$

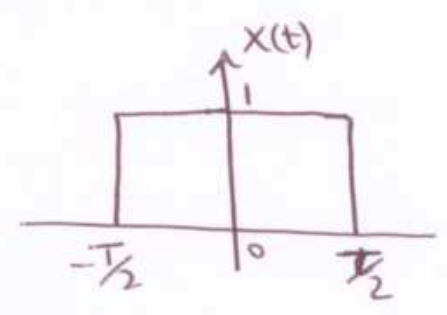
$$= \frac{1}{j\omega} \left[e^{j\omega T/2} - e^{-j\omega T/2} \right] \times \frac{2}{2}$$

$$= \frac{2}{\omega} \cdot \left[\frac{1}{2j} \left[e^{j\omega T/2} - e^{-j\omega T/2} \right] \right] \Rightarrow \sin(\omega \cdot T/2)$$

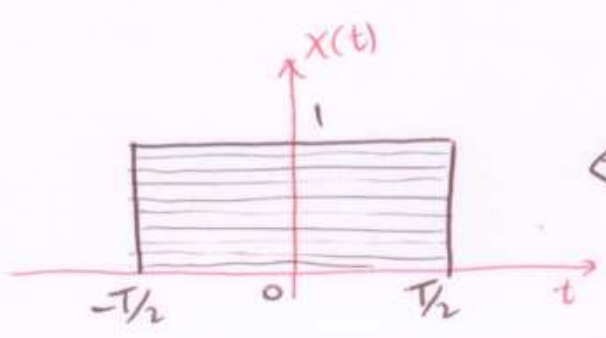
$$= \frac{2}{\omega} \cdot \sin(\omega \cdot T/2) \times \frac{T/2}{T/2} = T \cdot \frac{\sin(\omega \cdot T/2)}{\omega \cdot T/2}$$

$$\therefore X(\omega) = T \cdot \text{sinc}(\omega \cdot T/2)$$

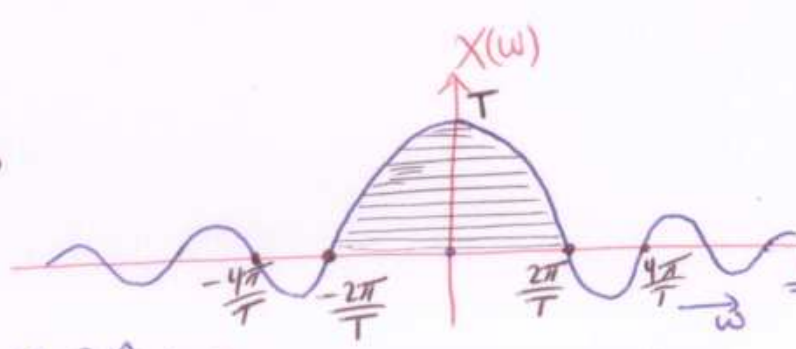
Notes $\text{sinc}(x) = \frac{\sin(x)}{x}$



Notes
 $\lim_{x \rightarrow 0} \text{sinc}(x)$
 $= \lim_{x \rightarrow 0} \frac{\sin x}{x} = \frac{0}{0}$ undefined
 $= \lim_{x \rightarrow 0} \frac{\cos x}{1} = 1$



$\Leftrightarrow$



$(-1, 1) \leftarrow \sin x$
 $\leftarrow X$
 زياده زياده ω زياده زياده ω

هذا الـ sinc

Ex 1

[6]

Find the Fourier transform for $x(t)$

$$x(t) = \begin{cases} e^{2t} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$X(\omega)$?!

$$\int_{-\infty}^{\infty} e^{2t} dt = \left. \frac{e^{2t}}{2} \right|_0^{\infty} = \infty - \frac{1}{2} = \infty \quad \underline{\text{div}}$$

there is no Fourier transform

But i can find the Laplace transform for it

as

$$X(s) = \frac{1}{s+2}$$

[3] Fourier transform of $x(t) = e^{-at} u(t)$

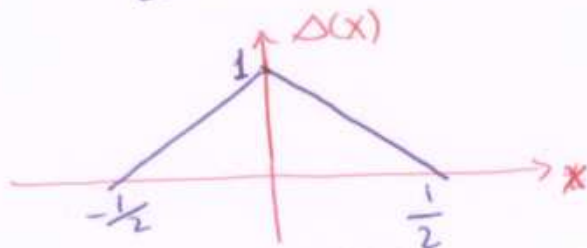
as shown before

$$x(t) \Leftrightarrow X(\omega)$$

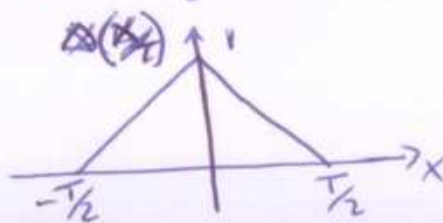
$$e^{-at} u(t) \Leftrightarrow \frac{1}{a+j\omega}$$

[4] Fourier transform of unit triangle function

$$\Delta(x) = \begin{cases} 0 & |x| \geq \frac{1}{2} \\ 1 - 2|x| & |x| < \frac{1}{2} \end{cases}$$

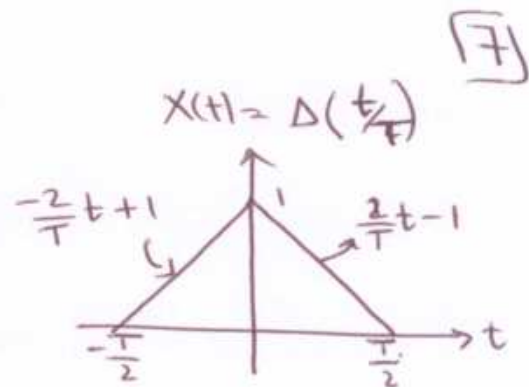


$$\Delta\left(\frac{x}{T}\right) \Rightarrow$$



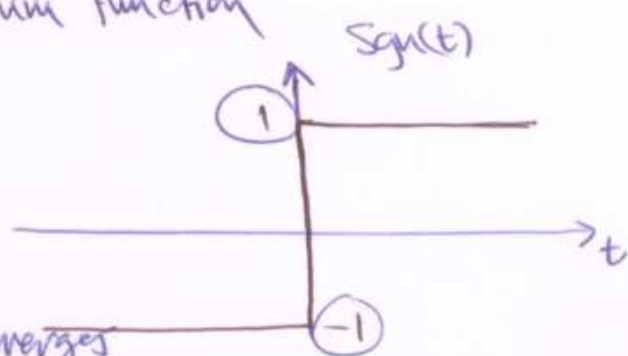
$$X(\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} dt$$

$$X(\omega) = \frac{T}{2} \text{sinc}^2\left(\frac{\omega T}{4}\right)$$



[5] Fourier Transform of Signum Function

$$\text{sgn}(t) = u(t) - u(-t)$$



the Fourier integral does not converge

because the integral of constant goes to ∞

So we use some approximation for this function

$$x = k \Rightarrow \int_{-\infty}^{\infty} k dt = kt \Big|_{-\infty}^{\infty} = \infty - (-\infty) = \infty \text{ div}$$

• way of approximating

$$\text{Let } \{ \text{sgn}(t) \} = \lim_{\epsilon \rightarrow 0} \left\{ \int_0^{\infty} e^{-\epsilon t} \cdot e^{-j\omega t} dt - \int_{-\infty}^0 e^{\epsilon t} \cdot e^{-j\omega t} dt \right\}$$

$$= \lim_{\epsilon \rightarrow 0} \left\{ \frac{e^{-(\epsilon + j\omega)t}}{-(\epsilon + j\omega)} \Big|_0^{\infty} - \frac{e^{(\epsilon - j\omega)t}}{(\epsilon - j\omega)} \Big|_{-\infty}^0 \right\}$$

$$= \lim_{\epsilon \rightarrow 0} \left\{ \frac{1}{(\epsilon + j\omega)} - \frac{1}{(\epsilon - j\omega)} \right\} = \lim_{\epsilon \rightarrow 0} \left\{ \frac{-2j\omega}{\epsilon^2 + \omega^2} \right\}$$

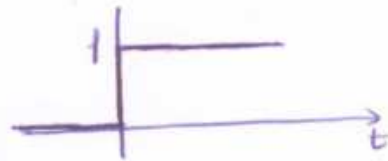
$$X(\omega) = \frac{-2j\omega}{\omega^2} = \frac{-2j}{\omega} = \left[\frac{2}{j\omega} \right]$$

$$\therefore \boxed{\text{sgn}(t) \Leftrightarrow \frac{2}{j\omega}}$$

8

Ex 2 Find the $\mathcal{F}\{x(t)\}$ where

$$x(t) = \frac{1}{2} + \frac{1}{2} \text{sgn}(t)$$



By table

$u(t) \rightarrow \frac{1}{j\omega}$

$$X(\omega) = \frac{1}{2} \cdot 2\pi \delta(\omega) + \frac{1}{2} \cdot \frac{2}{j\omega}$$

$$\boxed{X(\omega) = \pi \delta(\omega) + \frac{1}{j\omega}}$$

unit step

$\mathcal{F}\{u(t)\} = \frac{1}{j\omega}$

[6] Inverse Fourier transform of $\delta(\omega - \omega_0)$

By using sampling property of impulse

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\omega - \omega_0) \cdot e^{j\omega t} d\omega = \frac{1}{2\pi} e^{j\omega_0 t}$$

$$\frac{1}{2\pi} e^{j\omega_0 t} \xLeftrightarrow[\text{T.D.}] \delta(\omega - \omega_0)$$

$$e^{j\omega_0 t} \xLeftrightarrow[\text{F.T.}] 2\pi \delta(\omega - \omega_0)$$

$$e^{-j\omega_0 t} \xLeftrightarrow[\text{F.T.}] 2\pi \delta(\omega + \omega_0)$$

[7] Fourier transform of cos wot

[9]

Euler formula $\Rightarrow \cos wot = \frac{1}{2} [e^{jwot} + e^{-jwot}]$

$\therefore$ By the previous properties.

$$\cos wot \Leftrightarrow \pi [\delta(w-w_0) + \delta(w+w_0)]$$

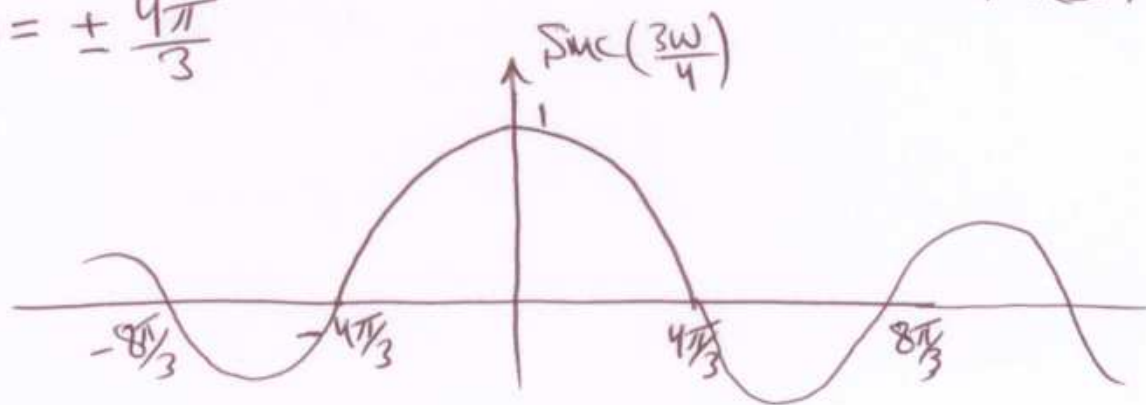
[8] Fourier transform of sin wot

$$\sin wot \Leftrightarrow \frac{\pi}{j} [\delta(w-w_0) - \delta(w+w_0)]$$

Notes:- Sketch the following function:-

1) $\text{Sinc}(\frac{3w}{4}) \Rightarrow \frac{\sin(\frac{3w}{4})}{\frac{3w}{4}} = 0 \text{ at } \frac{3w}{4} = \pm\pi$
because $\sin(\pm\pi) = 0$

$\therefore w = \pm \frac{4\pi}{3}$

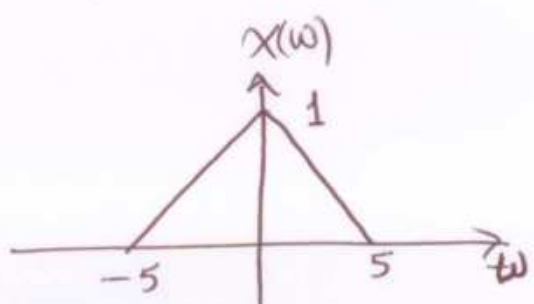


2) $\Delta(\frac{w}{10})$

$T = 10$ (length)

center $= 0$

$A = 1$



3)

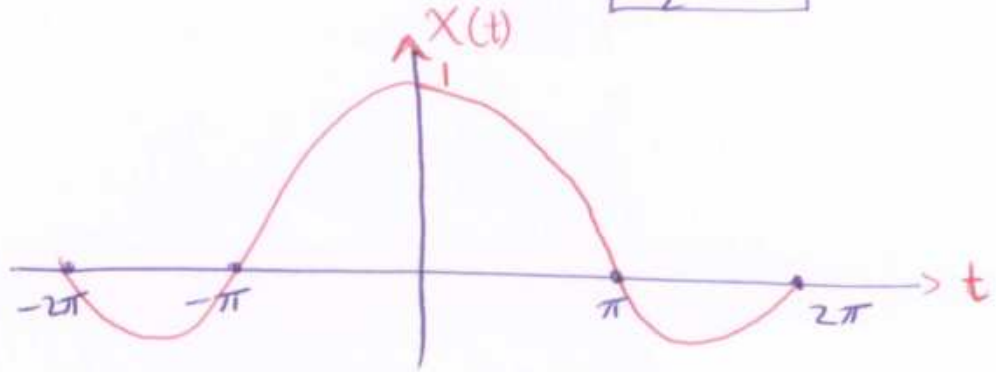
$$\text{sinc}(t) \cdot \text{rect}\left(\frac{t}{4\pi}\right)$$

like a window

$$t = \pm\pi \Rightarrow \text{sinc}(t) = 0$$

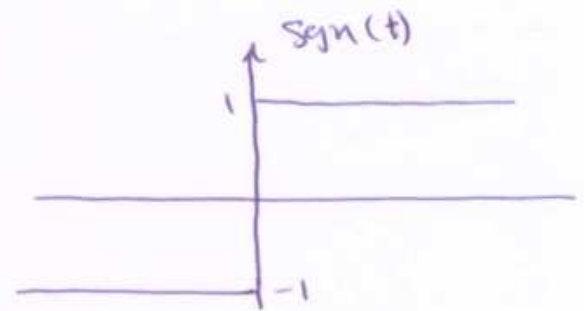
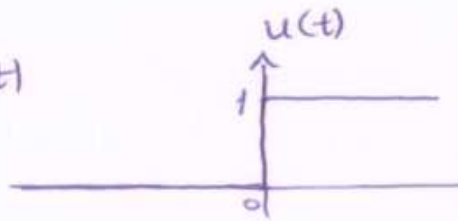
$$T = 4\pi \Rightarrow \text{length}$$

$$\boxed{\frac{T}{2} = 2\pi}$$



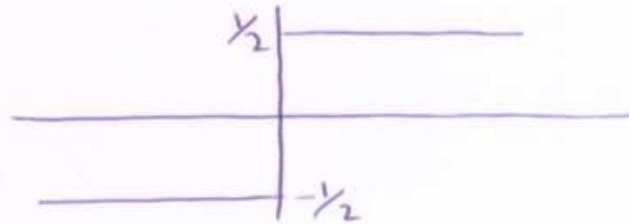
how?! the unit step $u(t) = \frac{1}{2} + \frac{1}{2} \text{sgn}(t)$

I know $u(t)$



$\Downarrow \times \frac{1}{2}$

$+\frac{1}{2}$
 $\Leftarrow$
shift to
up by $\frac{1}{2}$



$$u(t) = \frac{1}{2} + \frac{1}{2} \text{sgn}(t)$$

OK

7.3 Fourier Transform properties :-

از دو سیگنال (هر یک)

I) linearity :-

If

$$x_1(t) \longleftrightarrow X_1(\omega)$$

$$x_2(t) \longleftrightarrow X_2(\omega)$$

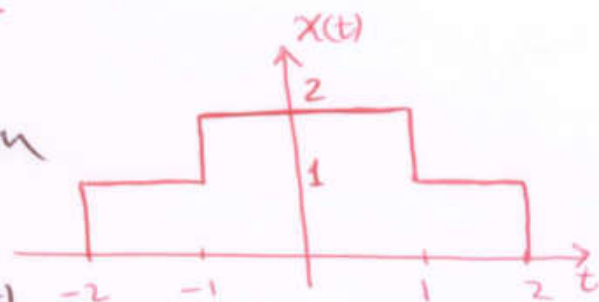
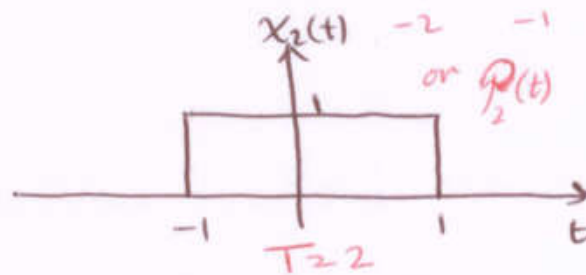
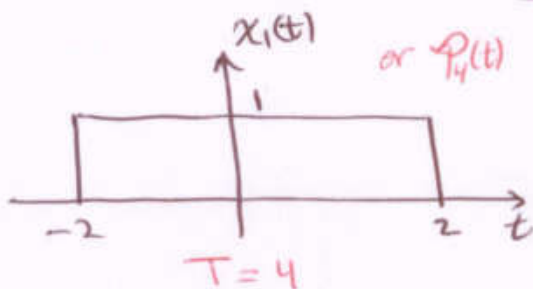
then

$$a_1 x_1(t) + a_2 x_2(t) \longleftrightarrow a_1 X_1(\omega) + a_2 X_2(\omega)$$

Example

Find the FT for the following signal

this signal come from summation between two rectangular signal



And we know the convert of rect from the table

$$\text{rect}\left(\frac{t}{T}\right) \longleftrightarrow T \cdot \text{sinc}\left(\omega \cdot \frac{T}{2}\right)$$

$$X(\omega) = X_1(\omega) + X_2(\omega)$$

$$X(\omega) = 4 \cdot \text{sinc}(2\omega) + 2 \text{sinc}(\omega)$$

#

[2] Time shifting

If $X(t) \longleftrightarrow X(\omega)$

then
time shift $\rightarrow X(t \pm t_0) \longleftrightarrow e^{\pm j\omega t_0} X(\omega)$

Frequency shift $\rightarrow X(t) \cdot e^{\pm j\omega_0 t} \longleftrightarrow X(\omega \pm \omega_0)$

Example

find the $\mathcal{F}\{x(t+1)\}$ if you have $X(\omega) = \text{rect}\left(\frac{(\omega-1)}{2}\right)$

$$X(t) \longleftrightarrow X(\omega)$$

$$X(t+1) \longleftrightarrow e^{j\omega} X(\omega) = e^{j\omega} X(\omega)$$

$$\therefore \mathcal{F}\{x(t+1)\} = e^{j\omega} \cdot \text{rect}\left(\frac{(\omega-1)}{2}\right) \quad \text{A.P.}$$

[3] Time scaling

If $X(t) \longleftrightarrow X(\omega)$

then $X(at) \longleftrightarrow \frac{1}{|a|} \cdot X\left(\frac{\omega}{a}\right)$

$\rightarrow$ where (a) is any real constant

Example find $\mathcal{F}\left\{a \cdot \text{rect}\left(a \frac{t}{T}\right)\right\}$

- we have the FT of $\text{rect}\left(\frac{t}{T}\right)$ in the table = $T \text{ sinc}(\omega \cdot \frac{T}{2})$

$$\therefore \mathcal{F}\left\{a \cdot \text{rect}\left(a \frac{t}{T}\right)\right\} = \frac{a}{|a|} \cdot T \text{ sinc}\left(\frac{\omega T}{2a}\right) \quad \text{A.P.}$$

Example 2 if $X(t) \longleftrightarrow X(\omega) = \text{rec}\left(\frac{\omega-1}{2}\right)$

find the $\mathcal{F}\{X(-2t+4)\}$

$X(t) \xleftarrow[\text{Scaling}]{\text{shifting}}$ $X(-2(t-2))$
 (Note: In the original image, arrows point from the operations to the expression. Shifting is associated with $(t-2)$ and Scaling with -2 .)

تحويل π shift الى اليمين 2 وحدة $\rightarrow$ $X(t-2)$

$$X(t) \longleftrightarrow X(\omega)$$

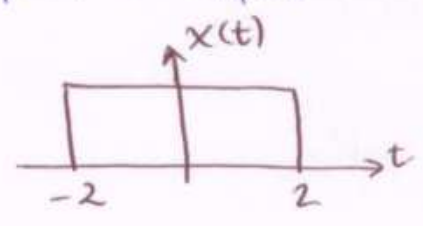
$$X(-2t) \longleftrightarrow \frac{1}{|-2|} X\left(\frac{\omega}{-2}\right) = \frac{1}{2} \text{rec}\left(\frac{\frac{\omega}{-2}-1}{2}\right)$$

$$X(-2(t-2)) \longleftrightarrow \frac{1}{|-2|} \cdot X\left(\frac{\omega}{-2}\right) \cdot e^{-j2\omega}$$

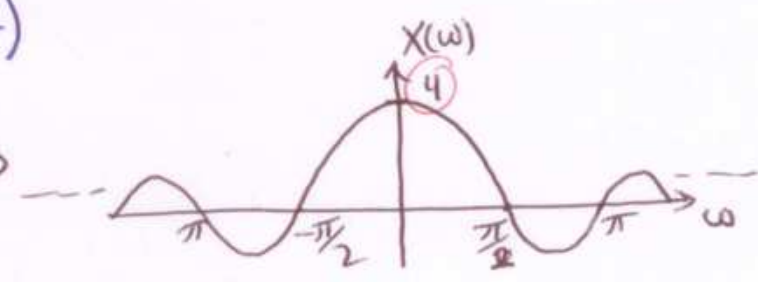
$$\therefore X(\omega) = \frac{1}{2} \cdot e^{-j2\omega} \cdot \text{rect}\left(\frac{\frac{\omega}{-2}-1}{2}\right) \quad \#$$

Notes: Time compression of a signal $X(at)$ result in its spectral expansion $X\left(\frac{\omega}{a}\right)$

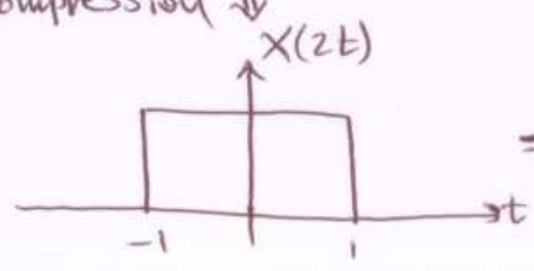
IP



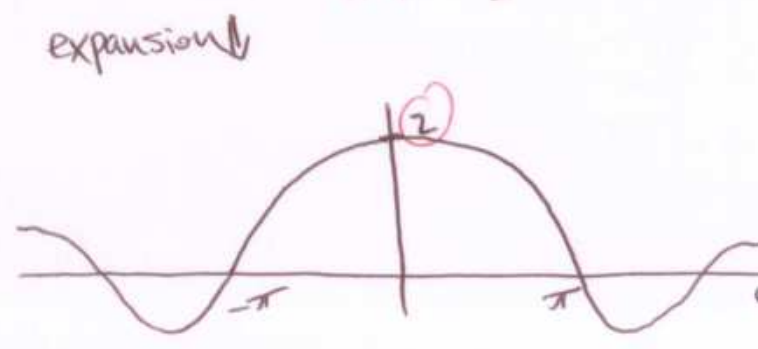
$\Rightarrow$



Compression $\Downarrow$



$\Rightarrow$



expansion $\Downarrow$

If $x(t) \longleftrightarrow X(\omega)$

then $\frac{dx(t)}{dt} \longleftrightarrow j\omega X(\omega)$

$$\frac{d^n x(t)}{dt^n} \longleftrightarrow (j\omega)^n X(\omega)$$

Ex 2 find the $\mathcal{F}\left\{\frac{dx(t)}{dt}\right\}$ & $\mathcal{F}\left\{\frac{d^2x(t)}{dt^2}\right\}$

if you have $X(\omega) = \text{rect}\left(\frac{\omega-1}{2}\right)$

- $\mathcal{F}\left\{\frac{dx(t)}{dt}\right\} = j\omega \cdot \text{rect}\left(\frac{\omega-1}{2}\right)$

- $\mathcal{F}\left\{\frac{d^2x(t)}{dt^2}\right\} = -\omega^2 \cdot \text{rect}\left(\frac{\omega-1}{2}\right)$

[5] Integration 2-

If $x(t) \longleftrightarrow X(\omega)$

then $\int_{-\infty}^t x(\tau) d\tau \longleftrightarrow \frac{X(\omega)}{j\omega} + \pi X(0) \delta(\omega)$

See the table of Fourier transform operations page 61 in lectures notes...

Example 2

15

By using time differentiation operation find the FT to

$$x(t) = \text{rect}\left(\frac{t}{T}\right)$$

- As we say before in chapter the differentiation of unit step is unit impulse $\frac{du(t)}{dt} = \delta(t)$

- As we know that the rect come from two unit step

$$\text{rect}\left(\frac{t}{T}\right) = u\left(t + \frac{T}{2}\right) - u\left(t - \frac{T}{2}\right)$$

So

$$\frac{dx(t)}{dt} = \delta\left(t + \frac{T}{2}\right) - \delta\left(t - \frac{T}{2}\right)$$

By using table

$$\delta(t) \longleftrightarrow 1$$

$$\delta\left(t + \frac{T}{2}\right) \longleftrightarrow 1 \cdot e^{j\omega \frac{T}{2}}$$

$$\frac{dx(t)}{dt} \longleftrightarrow (j\omega) X(\omega)$$

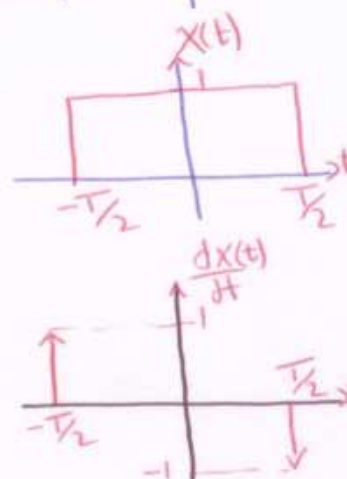
$$j\omega X(\omega) = e^{j\omega \frac{T}{2}} - e^{-j\omega \frac{T}{2}}$$

$$X(\omega) = \frac{1}{j\omega} \left[e^{j\omega \frac{T}{2}} - e^{-j\omega \frac{T}{2}} \right] \cdot \frac{2}{2}$$

$$X(\omega) = \frac{2}{\omega} \cdot \sin(\omega \frac{T}{2}) \cdot \frac{T/2}{T/2}$$

$$X(\omega) = T \text{sinc}(\omega \frac{T}{2})$$

#.



[6] Multiplication by a power (n) of t :-

[16]

$n =$ positive integer

If $X(t) \longleftrightarrow X(\omega)$

then $t^n X(t) \longleftrightarrow j^n \cdot \frac{d^n X(\omega)}{d\omega^n}$

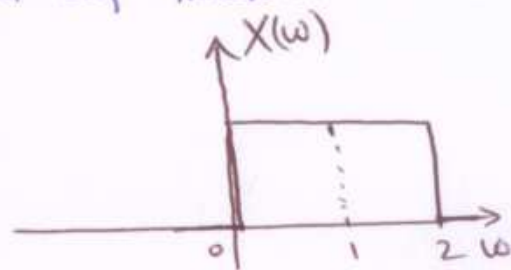
Example 2

Find the $\mathcal{F}\{t X(t)\}$ if $X(\omega) = \text{rect}\left(\frac{(\omega-1)}{2}\right)$

First, I don't know the differentiation of rect, so I need to sketch it & convert it to unit step function.

$$X(\omega) = u(\omega) - u(\omega-2)$$

$$\frac{dX(\omega)}{d\omega} = \delta(\omega) - \delta(\omega-2)$$



$$\mathcal{F}\{t X(t)\} = j [\delta(\omega) - \delta(\omega-2)]$$

[7] Convolution :-

If $X(t) \longleftrightarrow X(\omega)$

$h(t) \longleftrightarrow H(\omega)$

Then

duality
property

$$X(t) * h(t) \longleftrightarrow X(\omega) \cdot H(\omega)$$

Convolution in time

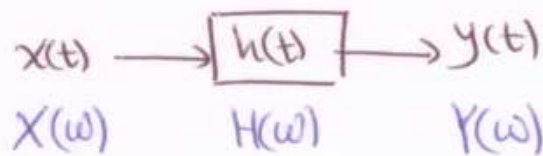
$$X(t) \cdot h(t) \longleftrightarrow \frac{1}{2\pi} X(\omega) * H(\omega)$$

Convolution in
Frequency

Example

Example if $x(t) = e^{-bt} u(t)$ & $h(t) = e^{-at} u(t)$

find $y(t)$?



(-jei jei jei hreef si wolo)

$$Y(\omega) = X(\omega) \cdot H(\omega)$$

$$X(\omega) = \frac{1}{j\omega + b}, \quad H(\omega) = \frac{1}{j\omega + a}$$

$$\therefore Y(w) = \left(\frac{1}{jw + b} \right) \cdot \left(\frac{1}{jw + a} \right)$$

غير موجود في اعداد
لذلك هو غير
Partial Fraction

$$Y(\omega) = \frac{k_1}{j\omega + b} \cdot \frac{k_2}{j\omega + a}$$

$$K_1 = \frac{1}{j\omega + a} \Big|_{j\omega = -b} = \frac{1}{a - b}$$

$$K_2 = \frac{1}{j\omega + b} \Big|_{j\omega = -a} = \frac{1}{b - a} = \frac{-1}{(a - b)}$$

$$\therefore y(t) = \frac{1}{a-b} \left[\frac{-bt}{e^t} - \frac{-at}{e^t} \right] \underline{\underline{u(t)}}$$

[8] Duality (دوالیت)

[18]

If $x(t) \longleftrightarrow X(\omega)$

then

$$X(t) \longleftrightarrow 2\pi \cdot x(-\omega)$$

یعنی اگر $x(t)$ و $X(\omega)$ دوال باشند، آنگاه $X(t)$ و $2\pi \cdot x(-\omega)$ دوال خواهند بود.

← این دو دوال در حوزه زمان و فرکانس قرار دارند. اگر $x(t)$ در حوزه زمان باشد، $X(\omega)$ در حوزه فرکانس خواهد بود.

یعنی اگر $x(t)$ در حوزه زمان باشد، $X(\omega)$ در حوزه فرکانس خواهد بود.

Ex2 Find the Fourier transform of $T \text{sinc}(t \cdot \frac{T}{2})$

In the table we have

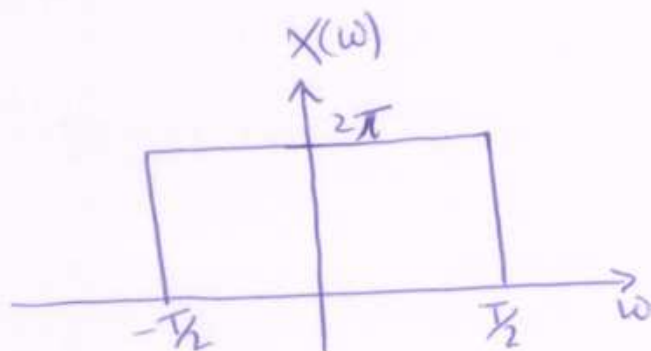
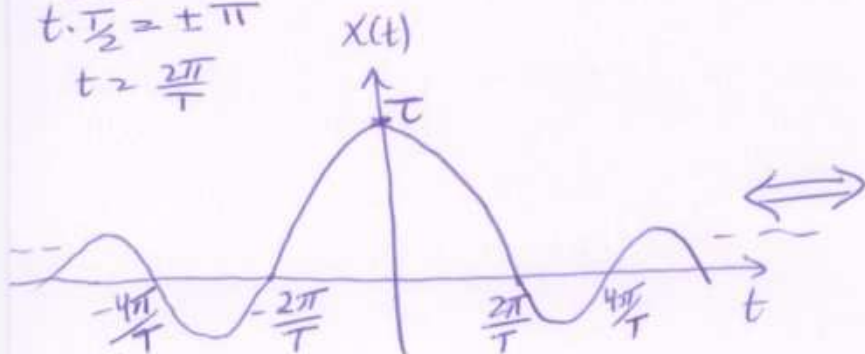
$$\text{rect}\left(\frac{t}{T}\right) \longleftrightarrow T \text{sinc}\left(\omega \frac{T}{2}\right)$$

Step 1: change $\omega \rightarrow t$
Step 2: change $t \rightarrow -\omega$
Multiply by 2π

$$T \text{sinc}\left(t \cdot \frac{T}{2}\right) \longleftrightarrow 2\pi \text{rect}\left(-\frac{\omega}{\frac{T}{2}}\right) = 2\pi \text{rect}\left(\frac{\omega}{\frac{T}{2}}\right)$$

$\text{rect}(-x) = \text{rect}(x)$ because is even function

Notes
 $t \cdot \frac{T}{2} = \pm \pi$
 $t = \pm \frac{2\pi}{T}$



✓

⊛ Examples-2

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a) Find the FT of $x(-2t+4)$ if you have

$$X(\omega) = \frac{1}{j\omega + 2}$$

$$x(t) \longleftrightarrow X(\omega)$$

$$x(-2t) \longleftrightarrow \frac{1}{|-2|} \cdot X\left(\frac{\omega}{-2}\right)$$

$$x(-2(t-2)) \longleftrightarrow \frac{1}{|-2|} \cdot e^{-j2\omega} \cdot X\left(\frac{\omega}{-2}\right)$$

$$\therefore X(\omega) = \frac{1}{2} \cdot e^{-j2\omega} \cdot \frac{1}{-j\frac{\omega}{2} + 2} \quad \text{R.}$$

b) Find the FT of $(t-2)x(t+1)$

linear combination $\Rightarrow t x(t+1) - 2x(t+1)$

$$x(t) \longleftrightarrow X(\omega)$$

multiplication by t $t x(t) \longleftrightarrow j \cdot \frac{d}{d\omega} X(\omega)$

$$x(t+1) \longleftrightarrow e^{-j\omega} \cdot X(\omega) = \frac{e^{-j\omega}}{j\omega + 2}$$

$$t x(t+1) \longleftrightarrow j \cdot \frac{d}{d\omega} \left[\frac{e^{-j\omega}}{j\omega + 2} \right] \quad \text{By calculator (complex)}$$

$$\therefore \mathcal{F}\{t(x(t+1)) - 2x(t+1)\} = j \cdot \frac{d}{d\omega} \left[\frac{e^{-j\omega}}{j\omega + 2} \right] - \frac{2 e^{-j\omega}}{j\omega + 2}$$

Notes 2

$$\left(\frac{d}{d\omega} \left[\frac{a\omega}{a\omega + b\omega} \right] = \frac{a(a\omega + b\omega) - a\omega(a+b)}{(a\omega + b\omega)^2} \right)$$

R.

$$c) \mathcal{F}\{e^{-j2t} \cdot x(2t)\}$$

$$x(t) \longleftrightarrow X(\omega)$$

$$x(2t) \longleftrightarrow \frac{1}{|2|} \cdot X\left(\frac{\omega}{2}\right)$$

$$e^{-j2t} x(2t) \longleftrightarrow \frac{1}{|2|} \cdot X\left(\frac{\omega+2}{2}\right)$$

$$X(\omega) = \frac{1}{2} \cdot \frac{1}{j\left(\frac{\omega+2}{2}\right) + 2} \quad \#$$

Ex2 Determine $X(\omega)$ if you have $x(t)$?

a) $x(t) = e^{-at} \cdot \cos \omega_0 t \cdot u(t)$
 $\uparrow \quad \quad \quad \uparrow$ I have its convert in the table

$$e^{-at} u(t) \longleftrightarrow \frac{1}{j\omega + a}$$

$$\cos \omega_0 t = \frac{1}{2} \left[e^{j\omega_0 t} + e^{-j\omega_0 t} \right]$$

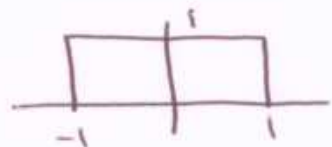
make shift in Freq Domain by opposite of sign $\pm \omega_0$

shift ω_0 $\uparrow$

$$\therefore x(t) = \frac{1}{2} e^{-at} \cdot e^{+j\omega_0 t} u(t) + \frac{1}{2} e^{-at} \cdot e^{-j\omega_0 t} u(t)$$

$$X(\omega) = \frac{1}{2} \cdot \frac{1}{a + j(\omega - \omega_0)} + \frac{1}{2} \cdot \frac{1}{a + j(\omega + \omega_0)} \quad \#$$

b) $x(t) = \cos^2 \pi t \left[\underbrace{u(t+1) - u(t-1)}_{\text{rect or } \mathcal{P}_2(t)} \right]$



From trigonometric properties

$$\cos^2 \pi t \longleftrightarrow \frac{1}{2} [1 + \cos 2\pi t]$$

$$x(t) = \frac{1}{2} [1 + \cos(2\pi t)] \cdot \text{rect}\left(\frac{t}{2}\right)$$

From table $\text{rect}(t/2) \longleftrightarrow 2 \text{sinc}(\omega)$

$$\cos 2\pi t = \frac{1}{2} \left[\underbrace{e^{j2\pi t}}_{\downarrow} + e^{-j2\pi t} \right]$$

(1) $\cos 2\pi t$ is a real signal, so its Fourier transform is real and even. $\uparrow$ shift in time

$$\therefore x(t) = \frac{1}{2} \text{rect}\left(\frac{t}{2}\right) + \frac{1}{4} \text{rect}\left[\frac{t}{2}\right] \cdot \underbrace{e^{j2\pi t}}_{\substack{\uparrow \\ \text{shift} \\ \text{in freq.}}} + \frac{1}{4} \text{rect}\left(\frac{t}{2}\right) \cdot e^{-j2\pi t}$$

$$X(\omega) = \frac{1}{2} \cdot 2 \text{sinc}(\omega) + \frac{1}{4} \cdot 2 \text{sinc}(\omega - 2\pi) + \frac{1}{4} \cdot 2 \text{sinc}(\omega + 2\pi)$$

$$X(\omega) = \text{sinc}(\omega) + \frac{1}{2} \text{sinc}(\omega - 2\pi) + \frac{1}{2} \text{sinc}(\omega + 2\pi)$$

$\mathcal{F}$

Ex: if you have $X(\omega) = e^{-a|\omega|}$, find $x(t)$

$$X(\omega) = \begin{cases} e^{-a\omega} & , \omega > 0 \\ e^{a\omega} & , \omega < 0 \end{cases}$$

$$\therefore X(\omega) = e^{-a\omega} u(\omega) + e^{a\omega} u(-\omega)$$

But I have in the table

$$e^{-at} u(t) \longleftrightarrow \frac{1}{a + j\omega}$$

By using duality

$$x(t) \longleftrightarrow X(\omega)$$

$$X(t) \longleftrightarrow 2\pi x(-\omega)$$

$\therefore$

$$e^{-at} u(t) \longleftrightarrow \frac{1}{a + j\omega}$$

$$\frac{1}{a + jt} \longleftrightarrow 2\pi e^{a\omega} u(-\omega)$$

$$\frac{1}{a - jt} \longleftrightarrow 2\pi e^{-a\omega} u(\omega)$$

$\therefore$

$$x(t) = \frac{1}{2\pi(a - jt)} + \frac{1}{2\pi(a + jt)}$$

Ans.

Ex2 if you have $X(\omega) = \text{sinc}(\omega - 10) + \text{sinc}(\omega + 10)$
find $x(t)$?

From the table $\text{rect}\left(\frac{t}{T}\right) \longleftrightarrow T \text{sinc}\left(\omega \frac{T}{2}\right)$

But I need $\text{sinc}(\omega) \Rightarrow$ so $\frac{T}{2} = 1 \Rightarrow T = 2$

$$\text{rect}\left(\frac{t}{2}\right) \longleftrightarrow 2 \text{sinc}(\omega)$$

$$\frac{1}{2} \text{rect}\left(\frac{t}{2}\right) \longleftrightarrow \text{sinc}(\omega)$$

* I know shift in frequency domain mean multiply by exp in time domain

$$e^{-j10t} \cdot \frac{1}{2} \text{rect}\left(\frac{t}{2}\right) \longleftrightarrow \text{sinc}(\omega - 10)$$

$$e^{j10t} \cdot \frac{1}{2} \text{rect}\left(\frac{t}{2}\right) \longleftrightarrow \text{sinc}(\omega + 10)$$

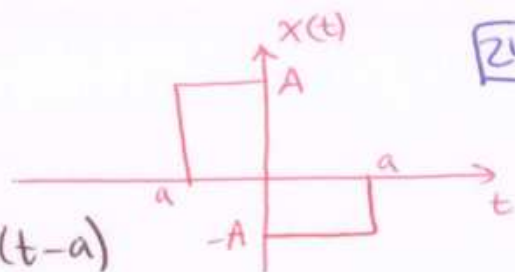
$$\therefore x(t) = \frac{1}{2} \text{rect}\left(\frac{t}{2}\right) \cdot e^{j10t} + \frac{1}{2} \text{rect}\left(\frac{t}{2}\right) \cdot e^{-j10t}$$

$$x(t) = \text{rect}\left(\frac{t}{2}\right) \cdot \frac{1}{2} [e^{j10t} + e^{-j10t}]$$

$x(t) = \text{rect}\left(\frac{t}{2}\right) \cdot \cos 10t$

#

Ex2 Find $X(\omega)$?



[24]

$$x(t) = A u(t+a) - 2A u(t) + A u(t-a)$$

From the table I know the convert of unit step.

$$u(t) \longleftrightarrow \frac{1}{j\omega} + \pi \delta(\omega)$$

$$u(t+a) \longleftrightarrow e^{j\omega a} \cdot \left(\frac{1}{j\omega} + \pi \delta(\omega) \right)$$

$$u(t-a) \longleftrightarrow e^{-j\omega a} \cdot \left(\frac{1}{j\omega} + \pi \delta(\omega) \right)$$

$$X(\omega) = A \left[\frac{1}{j\omega} + \pi \delta(\omega) \right] \cdot e^{j\omega a} - 2A \left[\frac{1}{j\omega} + \pi \delta(\omega) \right] + A \left[\frac{1}{j\omega} + \pi \delta(\omega) \right] \cdot e^{-j\omega a}$$

$$X(\omega) = \frac{A e^{j\omega a}}{j\omega} + A \pi e^{j\omega a} \delta(\omega) - \frac{2A}{j\omega} - 2A \pi \delta(\omega) + \frac{A e^{-j\omega a}}{j\omega} + A \pi e^{-j\omega a} \delta(\omega)$$

~~HL~~

[9] Modulation

is the process of varying one or more properties of a periodic wave form.

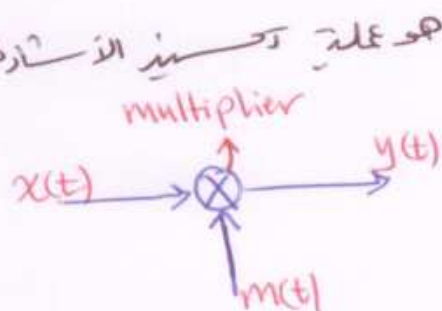
If

$$x(t) \longleftrightarrow X(\omega)$$

$$m(t) \longleftrightarrow M(\omega)$$

then

$$x(t) \cdot m(t) \longleftrightarrow \frac{1}{2\pi} [X(\omega) * M(\omega)]$$



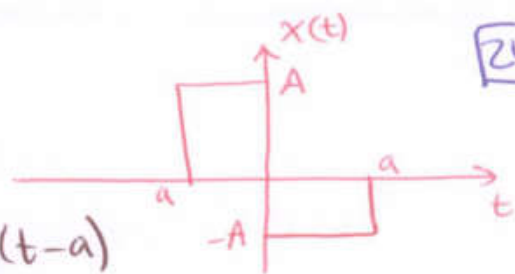
where

$x(t) \rightarrow$ modulating signal

$m(t) \rightarrow$ carrier signal ($\cos \omega t$)

$y(t) \rightarrow$ modulated signal

Ex 2 Find $X(\omega)$?



$$x(t) = A u(t+a) - 2A u(t) + A u(t-a)$$

From the table I know the convert of unit step.

$$u(t) \longleftrightarrow \frac{1}{j\omega} + \pi \delta(\omega)$$

$$u(t+a) \longleftrightarrow e^{j\omega a} \cdot \left(\frac{1}{j\omega} + \pi \delta(\omega) \right)$$

$$u(t-a) \longleftrightarrow e^{-j\omega a} \cdot \left(\frac{1}{j\omega} + \pi \delta(\omega) \right)$$

$$X(\omega) = A \left[\frac{1}{j\omega} + \pi \delta(\omega) \right] \cdot e^{j\omega a} - 2A \left[\frac{1}{j\omega} + \pi \delta(\omega) \right] + A \left[\frac{1}{j\omega} + \pi \delta(\omega) \right] \cdot e^{-j\omega a}$$

$$X(\omega) = \frac{A e^{j\omega a}}{j\omega} + A \pi e^{j\omega a} \delta(\omega) - \frac{2A}{j\omega} - 2A \pi \delta(\omega) + \frac{A e^{-j\omega a}}{j\omega} + A \pi e^{-j\omega a} \delta(\omega)$$

$\mathcal{H}$

[9] Modulation

is the process of varying one or more properties of a periodic wave form.

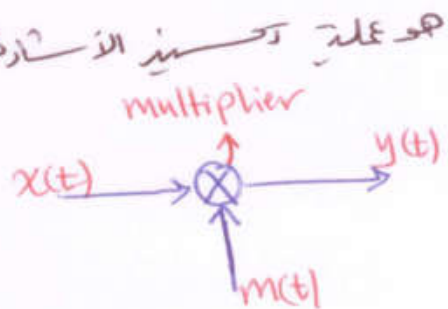
If

$$x(t) \longleftrightarrow X(\omega)$$

$$m(t) \longleftrightarrow M(\omega)$$

then

$$x(t) \cdot m(t) \longleftrightarrow \frac{1}{2\pi} [X(\omega) * M(\omega)]$$



where

$x(t) \rightarrow$ modulating signal

$m(t) \rightarrow$ carrier signal ($\cos \omega t$)

$y(t) \rightarrow$ modulated signal

* Application of the Fourier transform :-

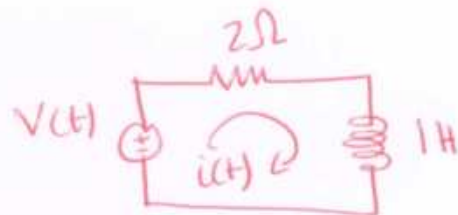
25

- Circuit Analysis
- Amplitude modulation
- Sampling Theorem
- Frequency multiplexing

I] Circuit Analysis

Ex: Find $i(t)$ for given

$$V(t) = 10e^{-t} u(t)$$



Notes

the order of system = # of storage element in the circuit

1st order system (RL) (RC) (R-L) (R-C) (R-L-C) (R-L-C)

time domain analysis (time domain) (time domain) (time domain)

freq-domain analysis (freq-domain) (freq-domain) (freq-domain)

Table of conversion

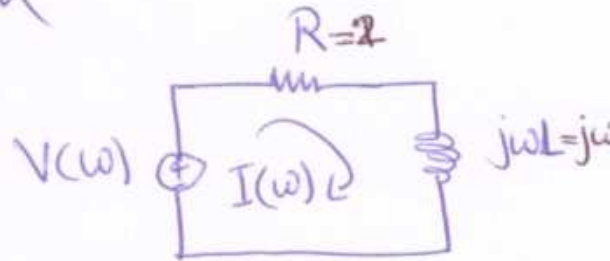
Component	Time	Frequency
Resistor	R	R
Inductor	L	$j\omega L$
Capacitor	C	$-j/\omega C$

So, the first step convert the circuit from time domain to frequency domain

$$V(\omega) = I(\omega) \cdot R + j\omega L I(\omega)$$

$$V(\omega) = I(\omega) (2 + j\omega)$$

$$I(\omega) = \frac{V(\omega)}{(2 + j\omega)}$$



$$V(t) \longleftrightarrow V(\omega)$$

$$e^{-t} u(t) \longleftrightarrow \frac{1}{(j\omega + 1)}$$

$$10 e^{-t} u(t) \longleftrightarrow \frac{10}{(j\omega + 1)}$$

$$I(\omega) = \frac{10}{(2 + j\omega)(1 + j\omega)} = \frac{10}{(2 + j\omega)(1 + j\omega)}$$

By partial fraction

$$I(\omega) = \frac{k_1}{(2 + j\omega)} + \frac{k_2}{(1 + j\omega)}$$

$$k_1 = \left. \frac{10}{j\omega + 1} \right|_{j\omega = -2} = -10$$

$$k_2 = \left. \frac{10}{j\omega + 2} \right|_{j\omega = -1} = 10$$

$$\therefore I(\omega) = \frac{-10}{2 + j\omega} + \frac{10}{(2 + j\omega)}$$

$$i(t) = -10 e^{-2t} u(t) + 10 e^{-t} u(t)$$

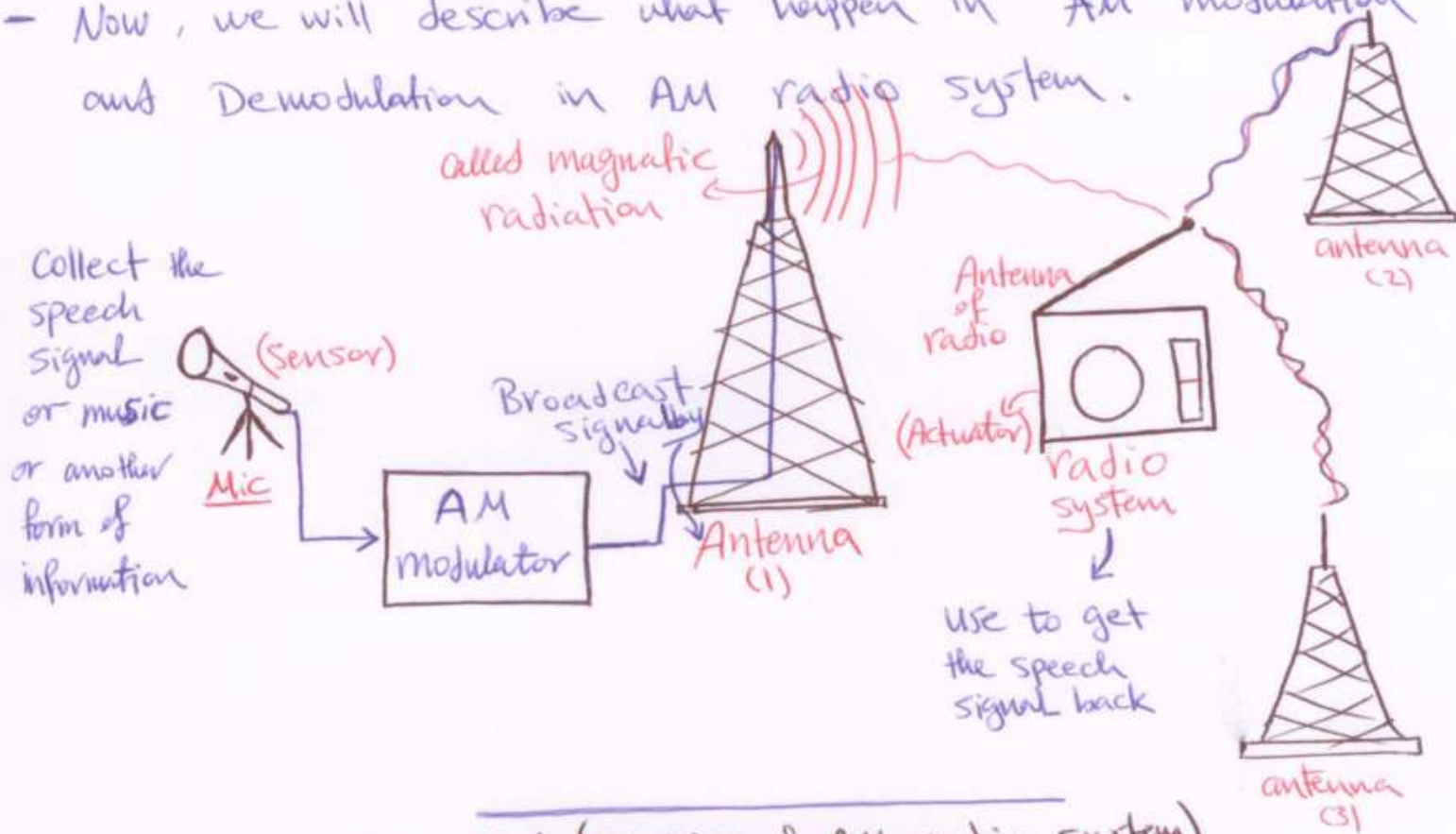
Q.E.D.

[2] Amplitude Modulation 2 (AM)

- Why we need to employing this type of conversion 2

- 1) To transmit information efficiently
- 2) To overcome hardware limitations
- 3) To reduce noise and interference $\rightarrow$ (noise)

- Now, we will describe what happen in AM modulation and Demodulation in AM radio system.

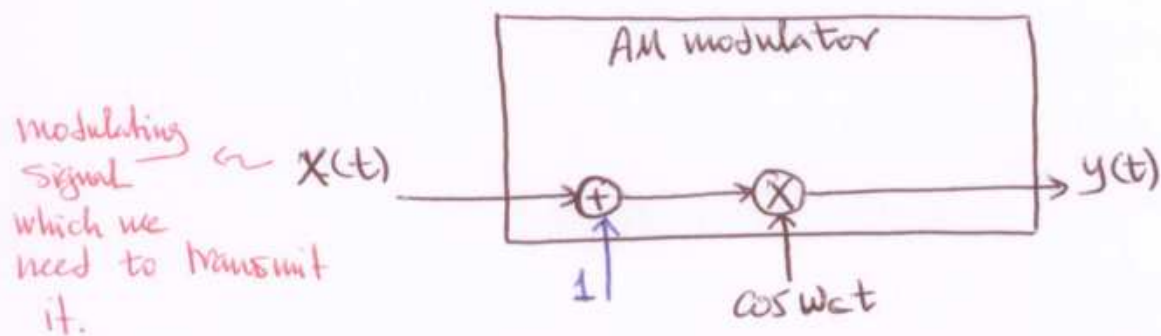


This called (Diagram of AM radio system)

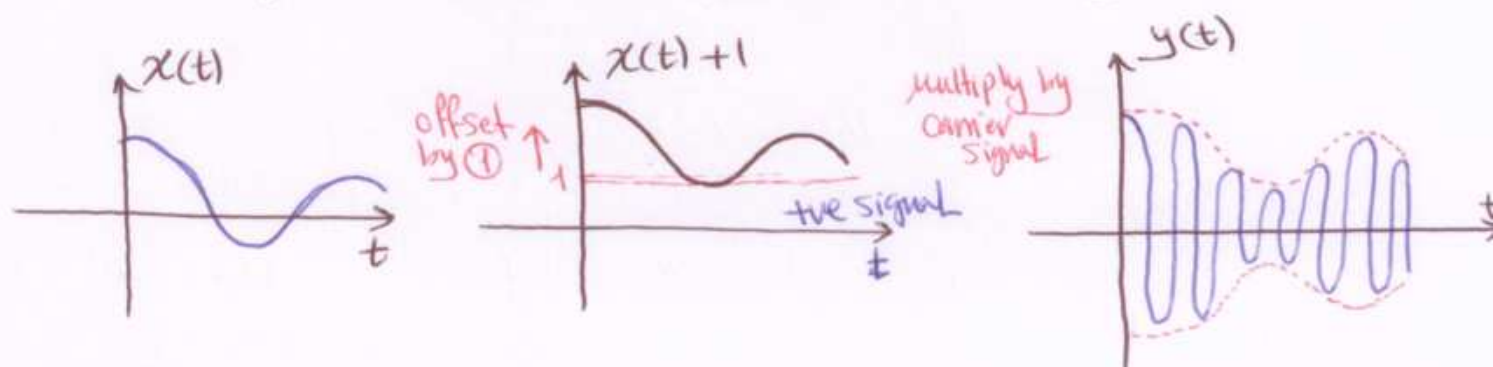
$\rightarrow$ The problem here how the radio will be select the signal we want from the another signal come from another antenna. but Broadcast another information.

(So, the main purpose of AM & DM modulation system is to solve this problem AS we will be see.)

→ First, let us see what happens in AM modulator 28



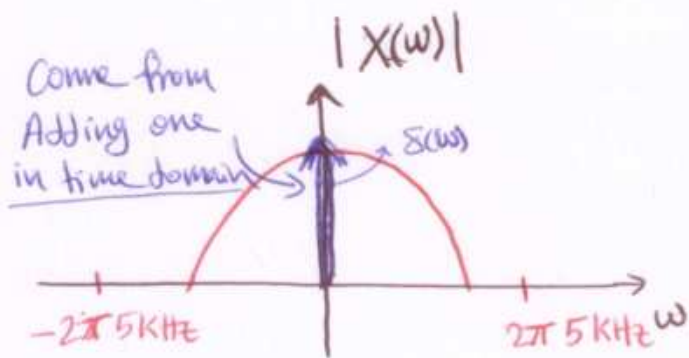
- The first step in AM modulator is Add one to input signal $X(t)$ to raise it up by one to cancel all negative part & doesn't go below the axis.
- Then multiply $(X(t)+1)$ by $\cos w_c t$ carrier signal with carrier frequency (w_c) .
- Then we have $y(t)$ modulated signal with very high power Amplifier and then we can transmit it or broadcast it by the antenna.
- what happens in each step: Assume any signal to be $X(t)$



$w_c \rightarrow$ carrier frequency.

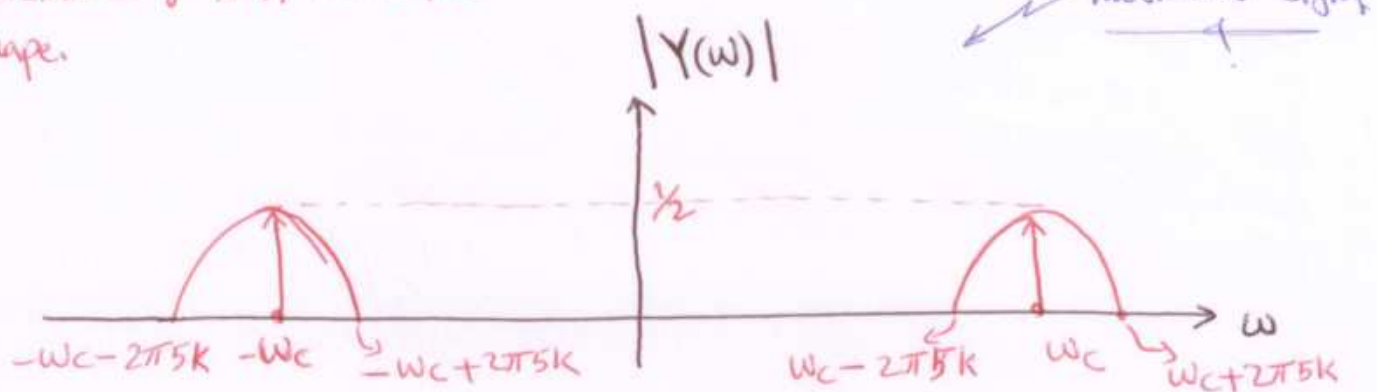
here the Amplitude of $\cos w_c t$ wave form dependent on the Amplitude of the signal $y(t)$
(that the reason is called)
AM

• Now, let we see what happen in frequency domain. 29



The Bandwidth of AM modulation
in KHz = 5KHz
finite or limited B.W

Assume that the magnitude spectrum of $x(t)$ have this shape.



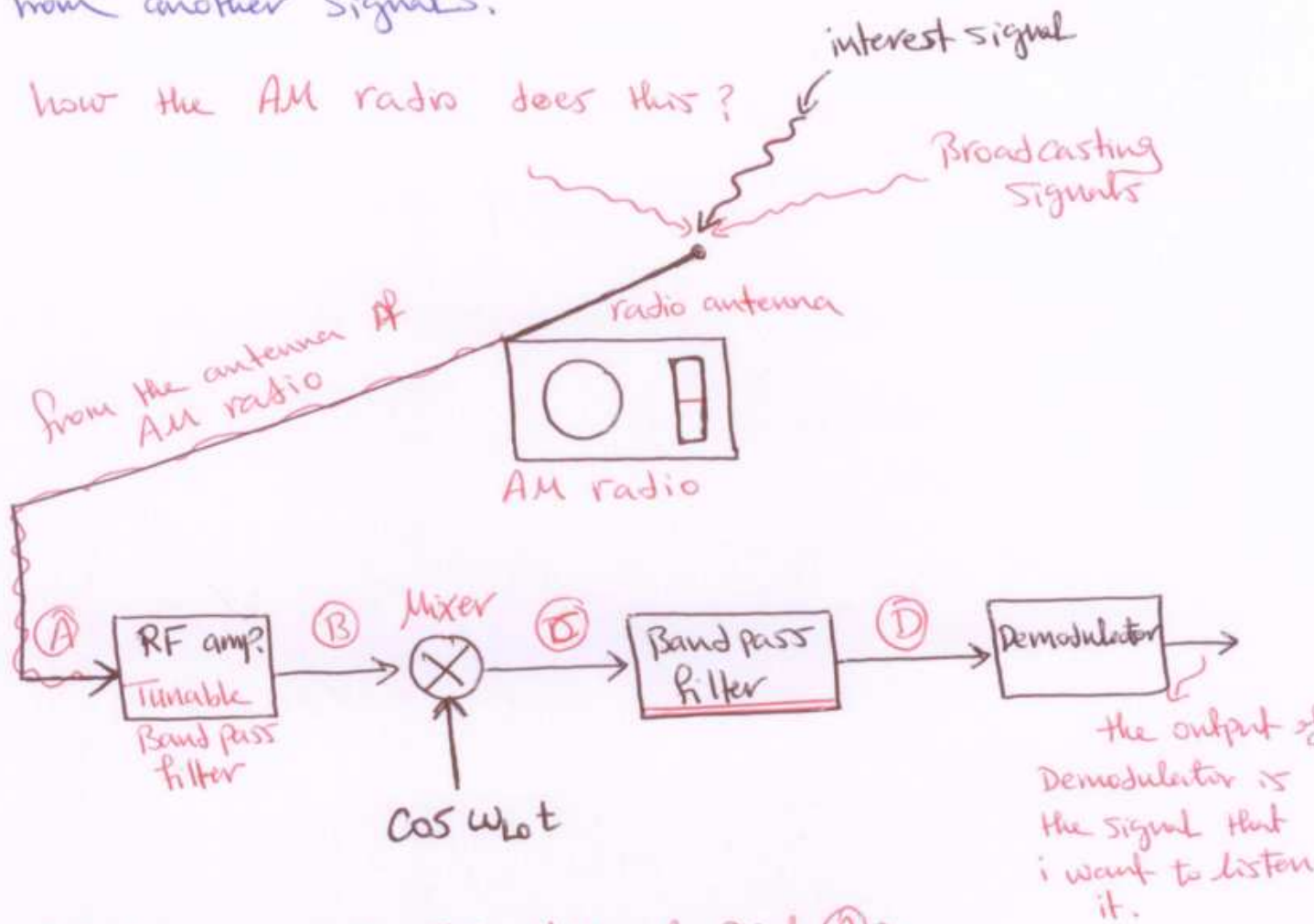
- we know the Fourier transform of $\cos \omega_c t$ which equal $\frac{1}{2} [\delta(\omega - \omega_c) + \delta(\omega + \omega_c)]$ which two shifted to ω_c & $-\omega_c$
- And when we multiply the $x(t)$ by $\cos \omega_c t$ in time domain the mean convolution in frequency domain that mean, we convolve $|X(\omega)|$ with $\cos \omega_c t$ that mean will be generate two copies of $|X(\omega)|$ shifted by ω_c & $-\omega_c$ because Delta function.
- Then we can Broadcast this AM modulated signal via antenna.

modulated signal $y(t)$

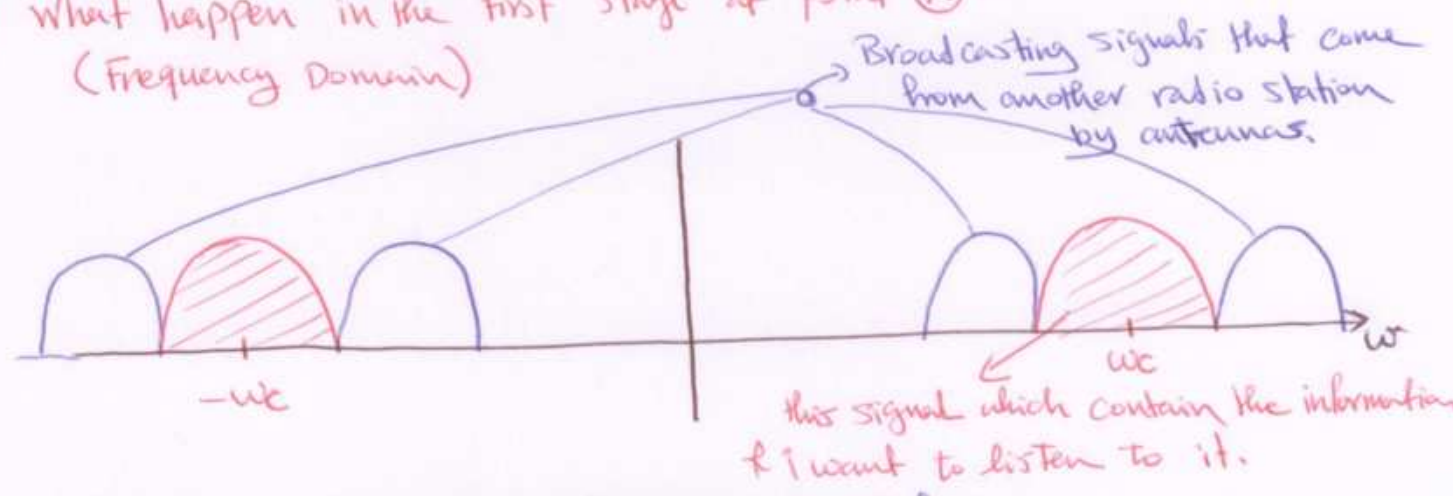
- Now, let we see what happen in Demodulation part

- After we broadcast our interest signal, the radio receiver take the signal we want and sperate it from another signals.

• how the AM radio does this?



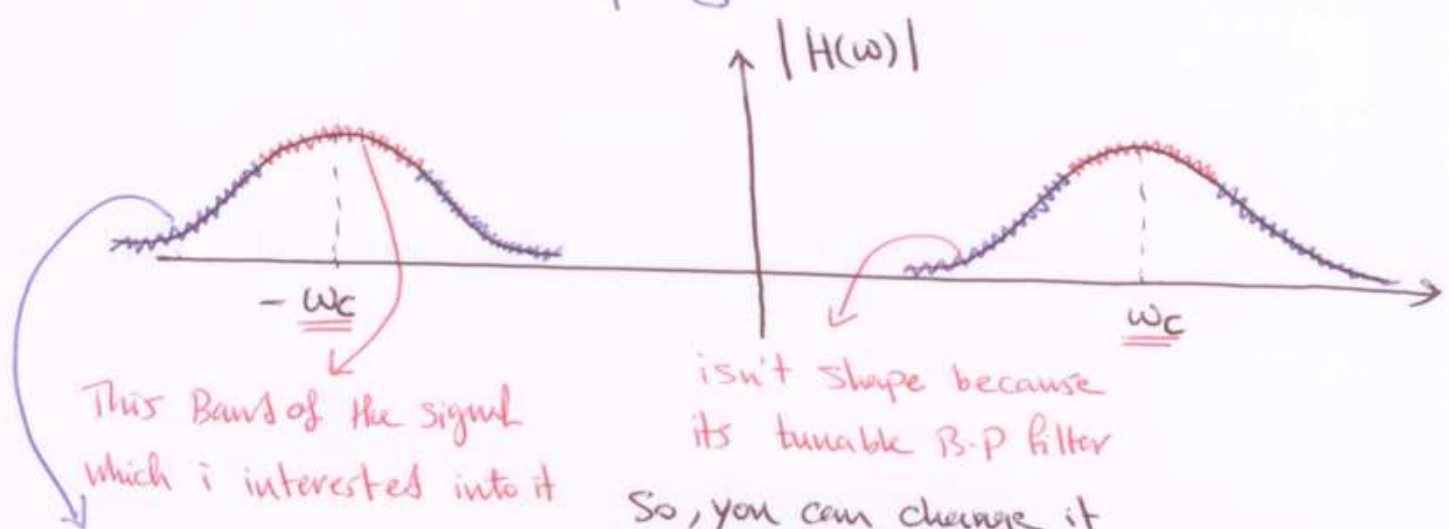
① What happen in the first stage of point ①? (Frequency Domain)



* RF amplifier - called radio frequency amplifier

↳ That takes the signal which come in from the antenna which some times very small and Amplified them.

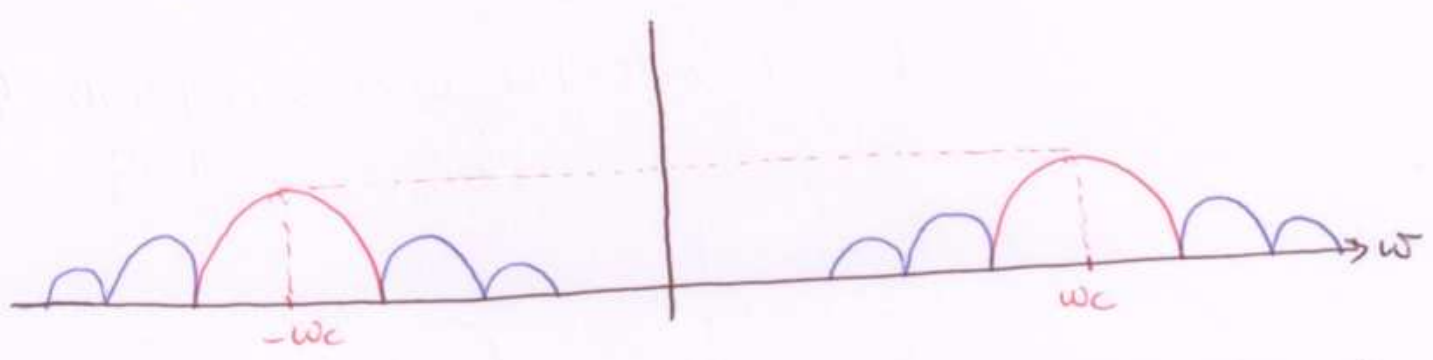
- then, also it pass from tunable Band pass filter.
which has this frequency spectrum.



isn't shape because
its tunable B-P filter
So, you can change it
center to the frequency of
station that you want to
listen to it.

another signals which
i don't interest with it
will attenuate by it
amplitude of B.P.F of
tunable

Ⓑ the output signal is the product signal with the
magnitude spectrum of tunable Band pass filter.



Now, this signal will input to mixer and multiply it
by $\cos \omega_c t$ in time domain, let we see what happen at Ⓒ

③ what is the mixer Does?

- is multiply my signal by $\cos \omega_{LO} t$ in time domain
which means convolution in frequency domain

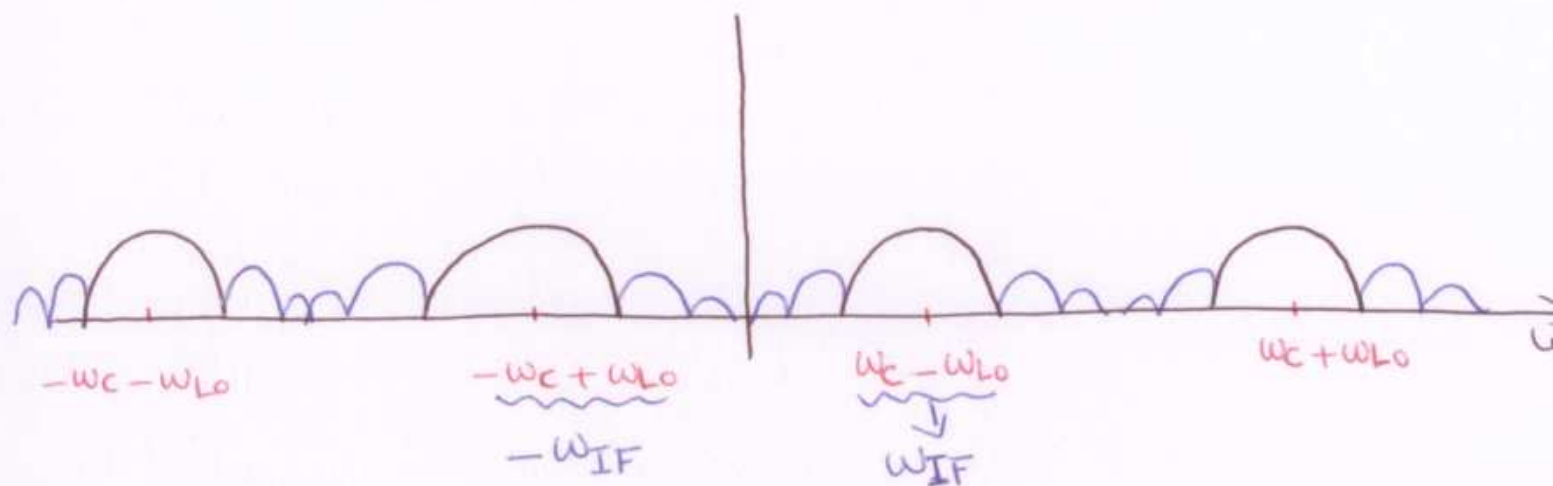
$\omega_{LO} \Rightarrow$ local oscillator frequency

$\hookrightarrow \omega_{LO} = \omega_c - \omega_{IF}$

ω_{IF} : intermediate frequency

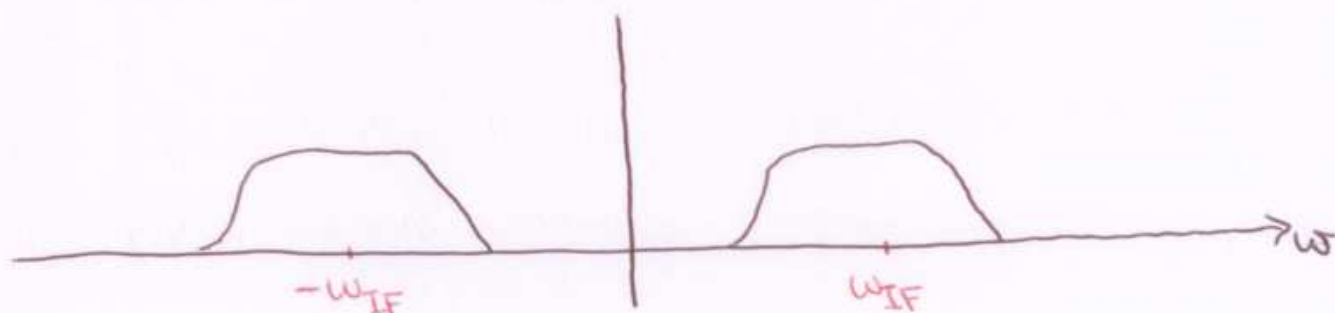
$\hookrightarrow$ which equal in AM radio sys

$2\pi 455 \text{ kHz}$

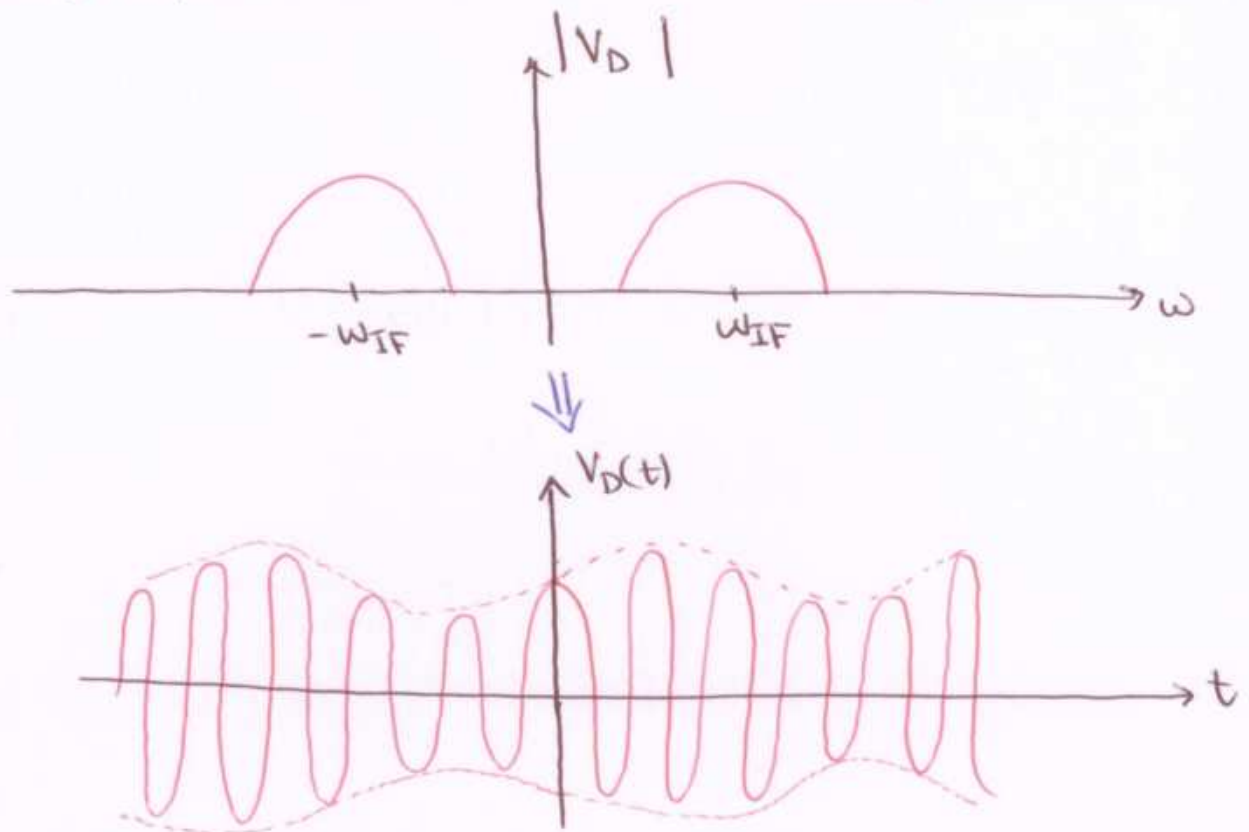


which is the center frequency of
the signal which i want to listen to it

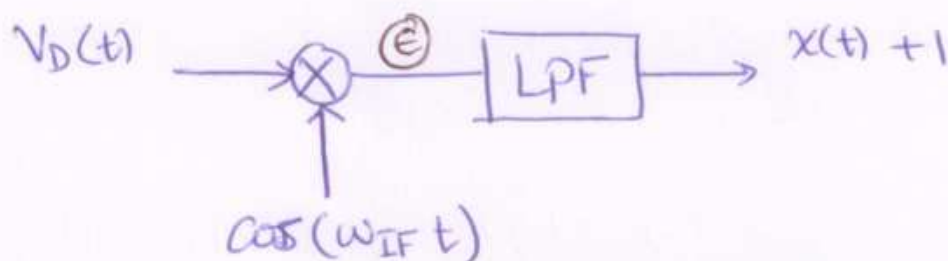
Now, this signal will input to the Band pass filter (shape filter)
which has this magnitude spectrum.



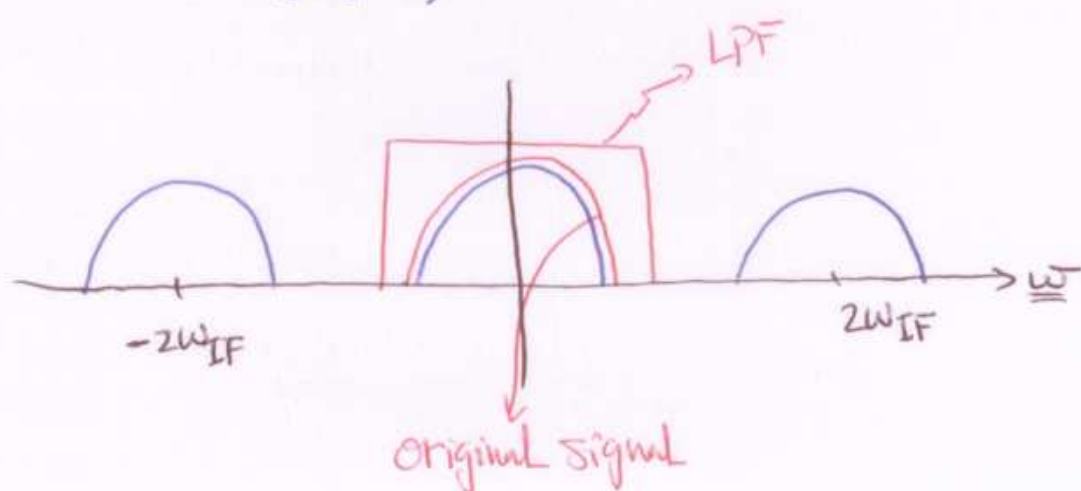
⑤ at this point, i have the demodulated signal



- Now, to get the transmit signal $(x(t)+1)$ i need to input it into the last stage



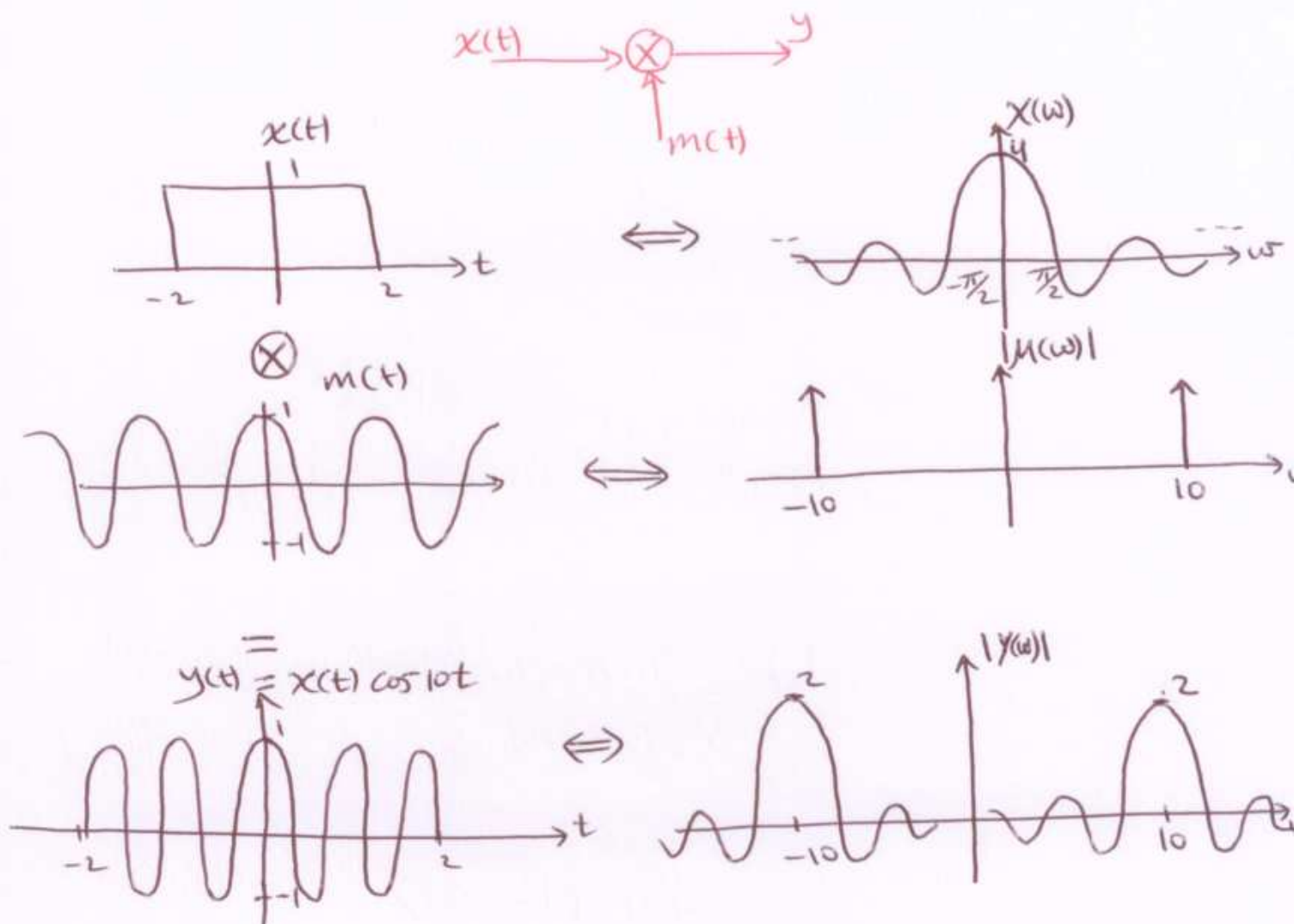
⑥



Example

(34)

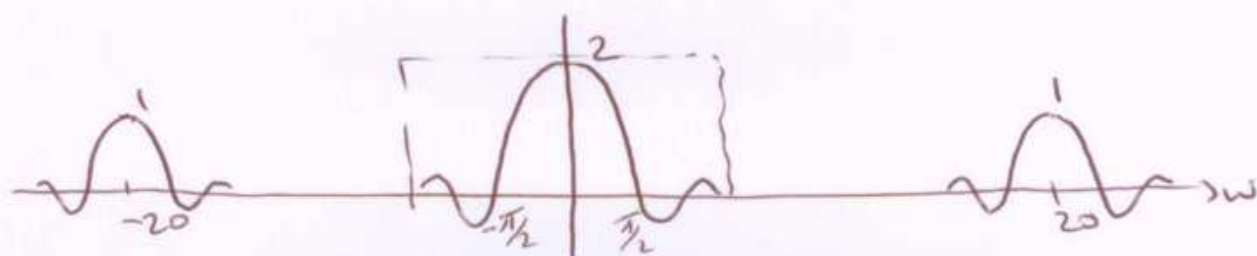
Find and sketch the Fourier transform of the signal where $x(t) = \text{rect}(t/4)$ & $m(t) = \cos 10t$



Demodulation
 $z = y(t) \cos 10t$

$$Z(\omega) = \frac{1}{2} [Y(\omega - \omega_0) + Y(\omega + \omega_0)] = \frac{1}{2} \left[\frac{1}{2} [X(\omega - \omega_0 - \omega_0) + X(\omega + \omega_0 + \omega_0)] + \frac{1}{2} [X(\omega + \omega_0 - \omega_0)] \right]$$

$$Z(\omega) = \frac{1}{4} X(\omega - 20) + \frac{1}{2} X(\omega) + \frac{1}{4} X(\omega + 20)$$



get in to LPF

[3] The Sampling Theorem 2 (A/D)

[35]

- Many signals originate as continuous time signals like music, voice ---
- we can sample the continuous signal by using equally spaced points in time (Δt), then we obtain a sequence of numbers.

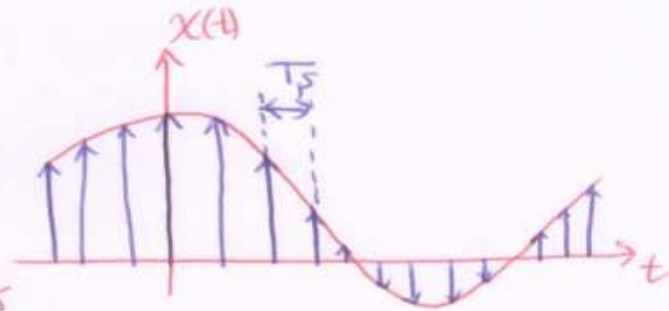
$$x(t) = x(\Delta n)$$

where n is integer numbers $n = 0, 1, 2, \dots$ or in -ve

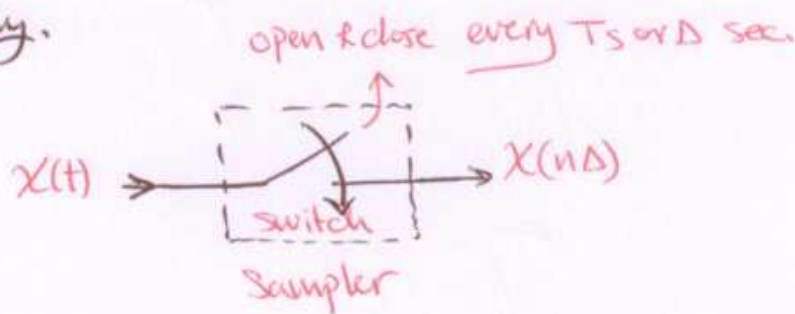
- Δt or Δ or T_s is called sampling period which is constant number in sec.

$$x(\Delta n) = x(t) \cdot \sum_{n=-\infty}^{\infty} \delta(t - \Delta n)$$

This called Train of pulses



→ Now, how we can sample a continuous signal practically.



$$x(t) \rightarrow \otimes \rightarrow y(t) = x(t) p(t)$$

$p(t)$

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - n\Delta)$$

Impulse modulation mode.

Shannon Sampling Theorem 2.

A continuous-time signal $x(t)$ can be uniquely reconstructed from its samples $x(\Delta n)$ with two conditions.

- 1) $x(t)$ must be band-limited with a maximum frequency ω_B
- 2) Sampling frequency ω_s of $x(\Delta n)$ or $x_s(t)$ must be greater than $2\omega_B$ (or equal)

$$\boxed{\omega_s \geq 2\omega_B} \quad \text{this condition called Nyquist criterion}$$

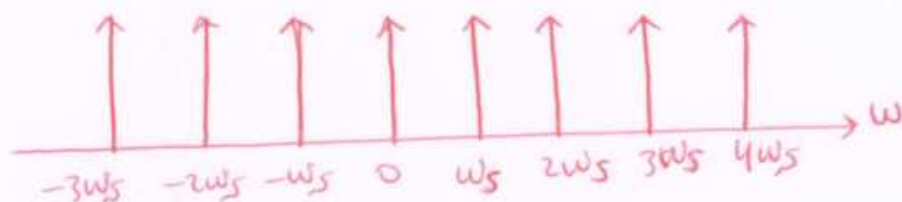
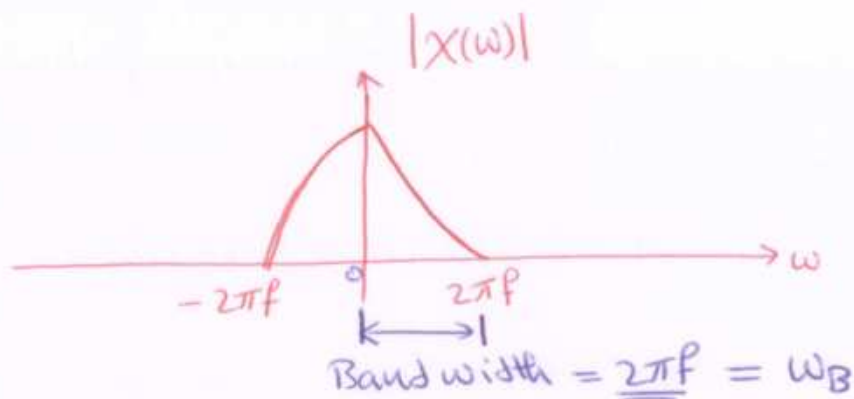
- ω_s called Nyquist frequency
- The Bandwidth (just take the positive frequency)

$$\boxed{B.W = f_{\max} - f_{\min}}$$

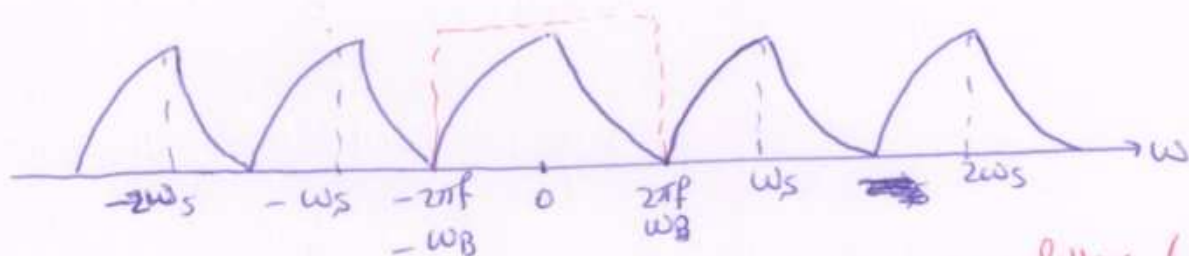
- if we have Low pass filter with maximum frequency $f_{\max}$, so the Bandwidth for it equal $B.W = f_{\max}$
- if we have Band pass filter with interval frequencies $[f_L, f_H]$ then the Bandwidth equal $B.W = f_H - f_L$

◦ Example 2

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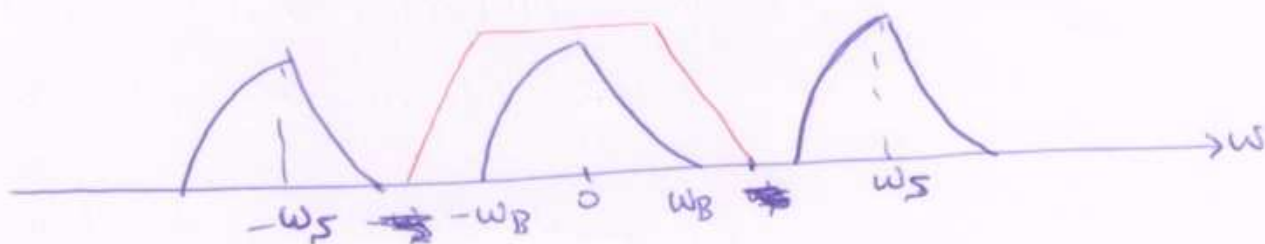


1] Let $\omega_S = 2\omega_B$



here we need very sharp low-pass filter (ideal) to be can reconstructed the original signal

2] $\omega_S > 2\omega_B$



3] $\omega_S < 2\omega_B$



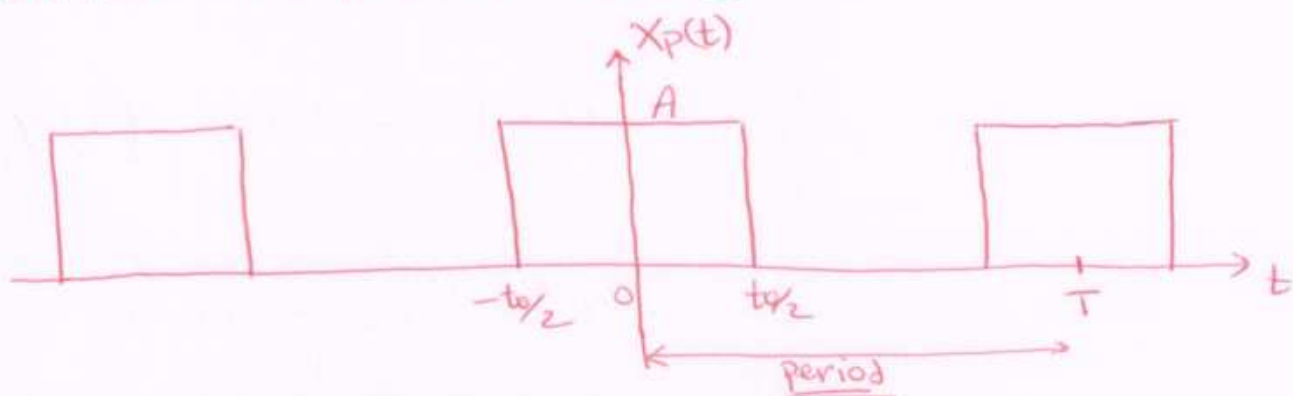
here we can't reconstruct the signal

* Connection Between F.S and F.T :-

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Example 2

Consider the periodic rectangular pulse train



$$T_0 = T \quad \& \quad \omega_0 = \frac{2\pi}{T}$$

$$x_p(t) = \begin{cases} A & -\tau/2 \leq t < \tau/2 \\ 0 & \text{otherwise} \end{cases}$$

$$c_n = \frac{1}{T} \int_{-\tau/2}^{\tau/2} A e^{-j\omega_0 n t} dt = \frac{A}{T} \cdot \frac{1}{-j\omega_0 n} \cdot e^{-j\omega_0 n t} \Big|_{-\tau/2}^{\tau/2}$$

$$= -\frac{A}{T} \cdot \frac{1}{j\omega_0 n} \left[e^{-j\omega_0 n \tau/2} - e^{j\omega_0 n \tau/2} \right]$$

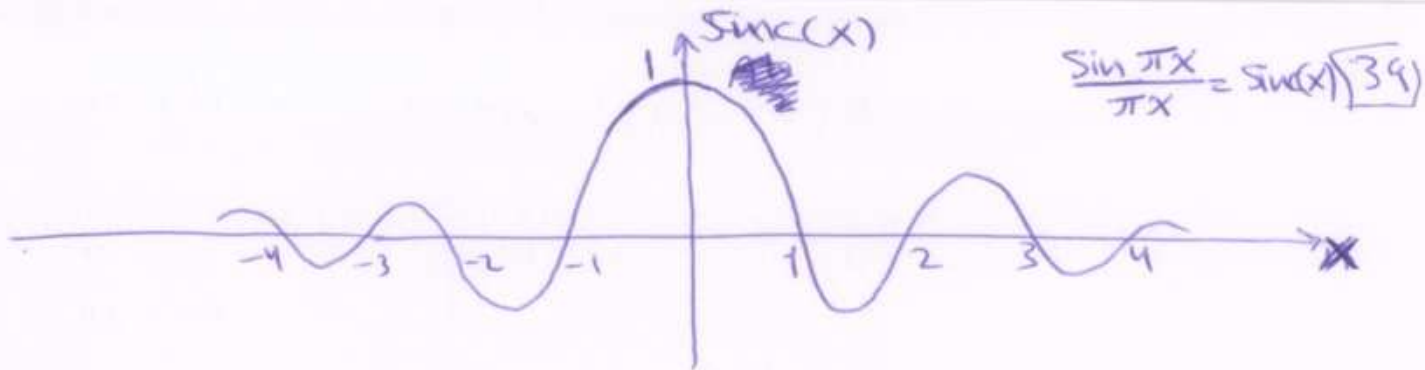
$$= \frac{A}{j\frac{2\pi}{T} \cdot n} \left[e^{+j\omega_0 n \tau/2} - e^{-j\omega_0 n \tau/2} \right]$$

$$= \frac{A}{\pi n} \sin(\omega_0 n \tau/2) \cdot \frac{\omega_0 \tau/2}{\omega_0 \tau/2}$$

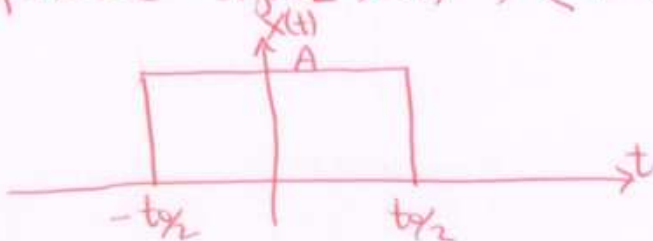
$$c_n = \frac{A \omega_0 \tau}{2\pi} \text{sinc}(\omega_0 n \tau/2)$$

$$\boxed{c_n = \frac{A \tau}{T} \text{sinc}(\omega_0 n \tau/2)}$$

$$\omega_0 n \tau/2 = \pm \pi$$



Now, if we have aperiodic signal $X(t)$, ($T \rightarrow \infty$)

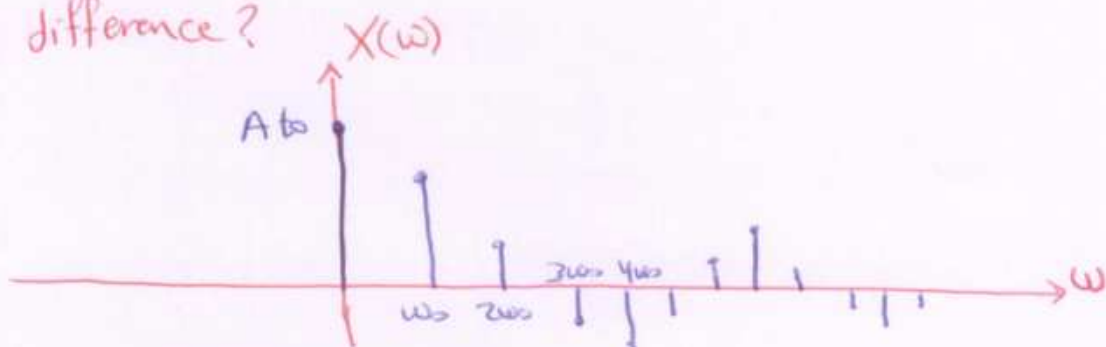


$$X(\omega) = T C_n = \lim_{T \rightarrow \infty} A \tau \text{sinc}(\omega \tau \frac{\tau}{2})$$

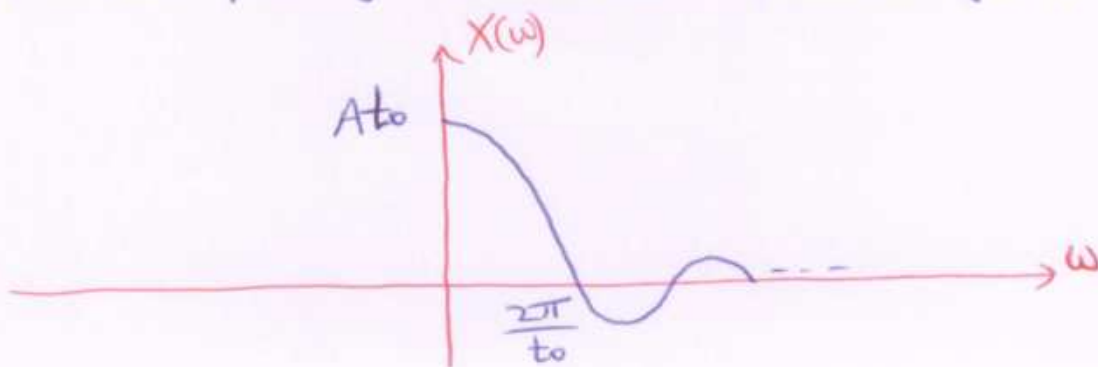
let $\boxed{\omega = \omega_0 \cdot n}$

$$\boxed{X(\omega) = A \tau \text{sinc}(\omega \cdot \frac{\tau}{2})}$$

• what is the difference?



- As we increase T your sample of sinc more ~~high~~
- as $T \rightarrow \infty$ you get continuous spectrum ($\omega_0 \rightarrow 0, \omega \rightarrow 0$)



- The Table of characteristic of signals in time domain and its response in frequency domain.

