

Signal & Linear system

Chapter 4 Frequency - Domain Analysis :
Laplace Transform

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4.1 The Laplace Transform

Motivation for the Laplace Transform

- ❑ CT Fourier transform enables us to do a lot of things, e.g.
 - Analyze frequency response of LTI systems
 - Sampling
 - Modulation
- ❑ Why do we need yet another transform?
- ❑ One view of Laplace Transform is as an *extension* of the Fourier transform to allow analysis of broader class of signals and systems
- ❑ In particular, Fourier transform *cannot* handle large (and important) classes of signals and *unstable* systems, i.e. when

$$\int_{-\infty}^{\infty} |x(t)| dt = \infty$$

4.1 The Laplace Transform

- The Laplace transform transforms the problem(D.EQ) from time domain to frequency domain.
- Then the solution of the original D.EQ is arrived at, by obtaining the inverse transforms.
- One of the problem that we faced using Fourier transform is many of the signals do not have Fourier transform.[ex. $\exp(\alpha t)u(t)$, $tu(t)$, and other time signals that are not absolutely integral]
- The difficulty could be resolved by extending the Fourier transform so that $x(t)$ is expressed as sum of complex exponentials, $\exp(-st)$ where $s = \sigma + j\omega$

4.1 The Laplace Transform

$\exp(-\sigma t)$, $\exp(\alpha t)u(t)$ satisfies the absolute integrable

For $\sigma > \alpha$

Ex. Given $x(t) = e^{\alpha t}u(t)$ Find the frequency domain

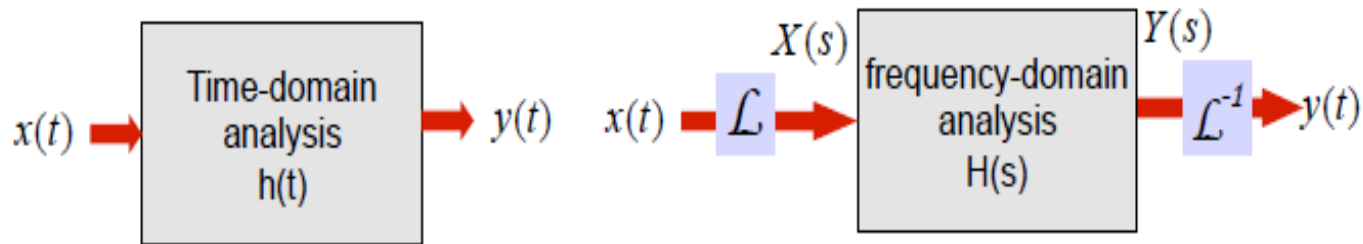
i. Fourier Transform $\int_{-\infty}^{\infty} e^{\alpha t} u(t) dt = \int_0^{\infty} e^{\alpha t} dt = \infty$

the Fourier transform does not exist.

ii. Laplace Transform $\int_{-\infty}^{\infty} e^{\alpha t} u(t) e^{-st} dt = \int_0^{\infty} e^{\alpha t} e^{-(\sigma+j\omega)t} dt$
 $= \int_0^{\infty} e^{-(\sigma-\alpha+j\omega)t} dt = \frac{1}{s-\alpha} \quad \sigma > \alpha$

4.1 The Laplace Transform

Laplace transform is the tool to map signals and system behavior from the time-domain into the frequency domain.



For a signal $x(t)$, its **Laplace transform is defined by:**

$$X_B(s) = L[x(t)] = \int_{-\infty}^{\infty} x(t)e^{-st} dt \quad \text{where } s = \sigma + j\omega$$

Bilateral

This general definite is known as **two-sided** (or **bilateral**) Laplace Transform.

4.1 The Laplace Transform

The one sided (unilateral) Laplace transform:

$$X(s) = L[x(t)] = \int_0^{\infty} x(t)e^{-st} dt$$

Ex Given $x(t) = e^{-bt}u(t)$ Find $X(s)$

Solution $X(s) = \int_0^{\infty} e^{-bt} e^{-st} dt = \int_0^{\infty} e^{-(s+b)t} dt$

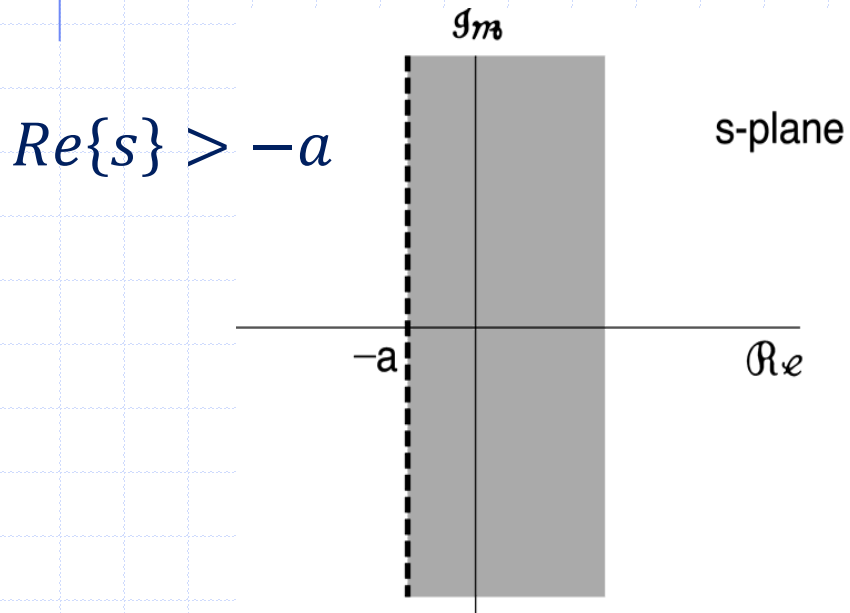
$$X(s) = \frac{1}{s+b} \quad \sigma > -b, \quad \text{Re}\{s\} > -b$$

The above signal has Laplace transform only if $\sigma > -b$ thus $X(s)$ exists only if $\text{Re}\{s\} > -b$

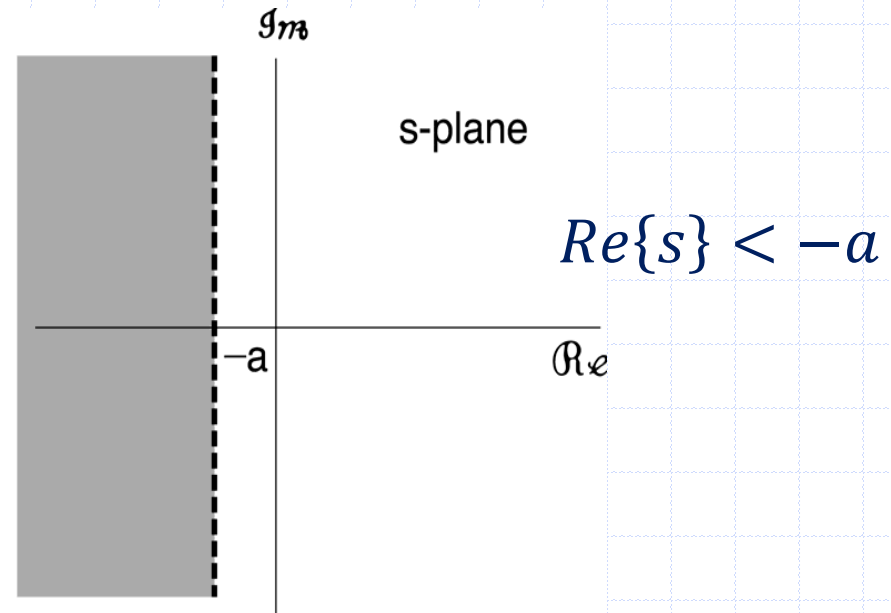
4.1 The Laplace Transform

The range of values for the complex variable S for which the Laplace transform converges is called the Region of Convergence (ROC)

$$x_1(t) = e^{-at}u(t)$$



$$x_2(t) = -e^{-at}u(-t)$$



4.1 The Laplace Transform

Ex. Given $x(t) = -e^{-bt}u(-t)$ Find $X(s)$

$$\text{Solution } X(s) = - \int_{-\infty}^0 e^{-bt} e^{-st} dt = \int_{-\infty}^0 e^{-(s+b)t} dt$$

$$X(s) = \frac{1}{s+b} \quad \sigma < -b, \quad \text{Re}\{s\} < -b$$

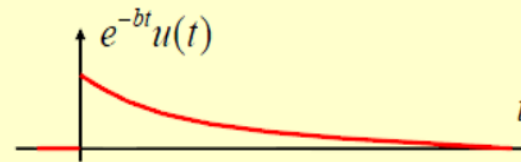
Note that $X(s)$ for the two previous examples are the same the only distinguish is ROC

Therefore, in order for the Laplace transform to be unique for each signal $x(t)$. The ROC must be specified as part of the transform

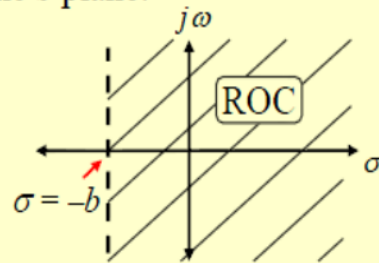
4.1 The Laplace Transform

If $b > 0$ then $x(t)$ itself decays:

For $b > 0$, $-b$ is negative



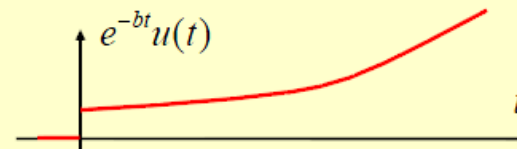
And we have on the s-plane:



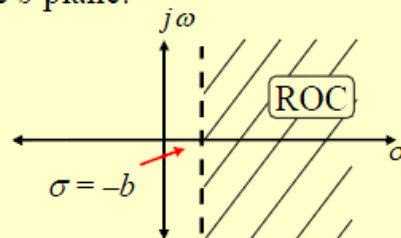
This case can be handled by the FT... and can also be handled by the LT

If $b < 0$ then $x(t)$ itself “explodes”:

For $b < 0$, $-b$ is positive



And we have on the s-plane:



This case can't be handled by the FT... but by restricting our focus to values of s in the ROC, the LT can handle it!!!

4.1 The Laplace Transform

Ex. Given $x(t) = 3e^{-2t}u(t) + 4e^t u(-t)$ Find $X(s)$

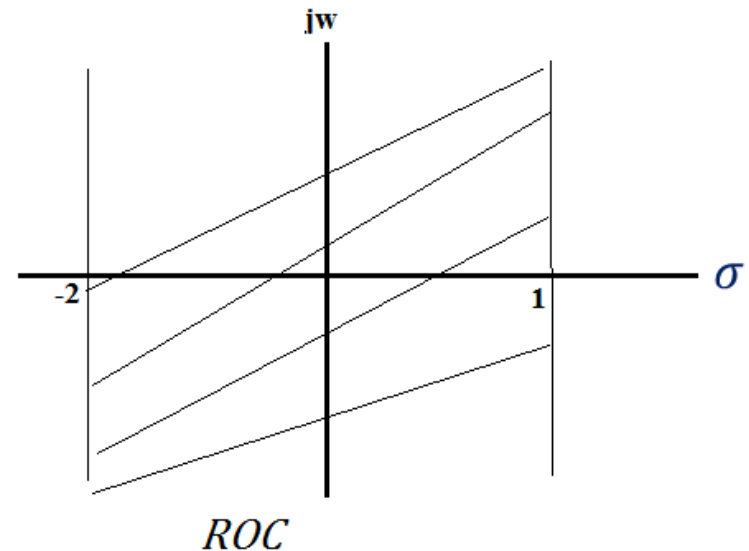
Solution $X(s) = \int_0^{\infty} 3e^{-2t} e^{-st} dt + \int_{-\infty}^0 4e^t e^{-st} dt$

$$X(s) = \frac{3}{s+2} - \frac{4}{s-1}$$

$$\text{Re}\{s\} > -2$$

$$\text{Re}\{s\} < 1$$

$$-2 < \text{Re}\{s\} < 1$$



4.1 The Laplace Transform

Ex. Given $x(t) = e^{2t}u(t)$ Find $X(s)$

Solution $X(s) = \int_0^{\infty} e^{2t} e^{-st} dt = \int_0^{\infty} e^{-(s-2)t} dt$

$$X(s) = \frac{1}{s-2}; \quad \operatorname{Re}\{s\} > 2$$

Ex. Find the Laplace transform of $\delta(t)$ and $u(t)$.

$$\mathcal{L}[\delta(t)] = \int_0^{\infty} \delta(t) e^{-st} dt = 1 \quad \text{for all } s$$

$$\mathcal{L}[\delta(t)] \Leftrightarrow 1$$

$$\begin{aligned} \mathcal{L}[u(t)] &= \int_0^{\infty} u(t) e^{-st} dt = \int_0^{\infty} e^{-st} dt \\ &= -\frac{1}{s} e^{-st} \Big|_0^{\infty} = \frac{1}{s} \quad \operatorname{Re} s > 0 \end{aligned}$$

$$\mathcal{L}[u(t)] \Leftrightarrow \frac{1}{s}$$

4.1 The Laplace Transform

Ex. Find the Laplace transform of $e^{at}u(t)$ and $\cos \omega_0 t u(t)$.

$$\begin{aligned}\mathcal{L}[e^{at}u(t)] &= \int_0^{\infty} e^{at} e^{-st} dt \\ &= \int_0^{\infty} e^{-(s-a)t} dt = \frac{1}{s-a}\end{aligned}$$

$$\mathcal{L}[e^{at}u(t)] \Leftrightarrow \frac{1}{s-a}$$

$$\begin{aligned}\mathcal{L}[\cos \omega_0 t u(t)] &= \frac{1}{2} \mathcal{L}[e^{j\omega_0 t} u(t) + e^{-j\omega_0 t} u(t)] \\ &= \frac{1}{2} \left[\frac{1}{s-j\omega_0} + \frac{1}{s+j\omega_0} \right] = \frac{s}{s^2 + \omega_0^2}\end{aligned}$$

$$\mathcal{L}[\cos \omega_0 t u(t)] \Leftrightarrow \frac{s}{s^2 + \omega_0^2}$$

Table of Laplace Transforms

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$	$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1. 1	$\frac{1}{s}$	2. e^{at}	$\frac{1}{s-a}$
3. $t^n, n=1,2,3,\dots$	$\frac{n!}{s^{n+1}}$	4. $t^p, p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}$
5. $\sqrt{t}$	$\frac{\sqrt{\pi}}{2s^{3/2}}$	6. $t^{n-1/2}, n=1,2,3,\dots$	$\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)\sqrt{\pi}}{2^n s^{n+1/2}}$
7. $\sin(at)$	$\frac{a}{s^2+a^2}$	8. $\cos(at)$	$\frac{s}{s^2+a^2}$
9. $t \sin(at)$	$\frac{2as}{(s^2+a^2)^2}$	10. $t \cos(at)$	$\frac{s^2-a^2}{(s^2+a^2)^2}$
11. $\sin(at) - at \cos(at)$	$\frac{2a^2}{(s^2+a^2)^2}$	12. $\sin(at) + at \cos(at)$	$\frac{2as^2}{(s^2+a^2)^2}$
13. $\cos(at) - at \sin(at)$	$\frac{s(s^2-a^2)}{(s^2+a^2)^2}$	14. $\cos(at) + at \sin(at)$	$\frac{s(s^2+3a^2)}{(s^2+a^2)^2}$
15. $\sin(at+b)$	$\frac{s \sin(b) + a \cos(b)}{s^2+a^2}$	16. $\cos(at+b)$	$\frac{s \cos(b) - a \sin(b)}{s^2+a^2}$
17. $\sinh(at)$	$\frac{a}{s^2-a^2}$	18. $\cosh(at)$	$\frac{s}{s^2-a^2}$
19. $e^{at} \sin(bt)$	$\frac{b}{(s-a)^2+b^2}$	20. $e^{at} \cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$
21. $e^{at} \sinh(bt)$	$\frac{b}{(s-a)^2-b^2}$	22. $e^{at} \cosh(bt)$	$\frac{s-a}{(s-a)^2-b^2}$
23. $t^n e^{at}, n=1,2,3,\dots$	$\frac{n!}{(s-a)^{n+1}}$	24. $f(ct)$	$\frac{1}{c} F\left(\frac{s}{c}\right)$
25. $u_c(t) = u(t-c)$ Heaviside Function	$\frac{e^{-as}}{s}$	26. $\delta(t-c)$ Dirac Delta Function	e^{-as}
27. $u_c(t) f(t-c)$	$e^{-as} F(s)$	28. $u_c(t) g(t)$	$e^{-as} \mathcal{L}\{g(t+c)\}$
29. $e^{at} f(t)$	$F(s-a)$	30. $t^n f(t), n=1,2,3,\dots$	$(-1)^n F^{(n)}(s)$
31. $\frac{1}{t} f(t)$	$\int_s^\infty F(u) du$	32. $\int_0^t f(v) dv$	$\frac{F(s)}{s}$
33. $\int_0^t f(t-\tau) g(\tau) d\tau$	$F(s) G(s)$	34. $f(t+T) - f(t)$	$\frac{\int_0^T e^{-s\tau} f(\tau) d\tau}{1-e^{-sT}}$
35. $f'(t)$	$sF(s) - f(0)$	36. $f''(t)$	$s^2 F(s) - sf'(0) - f''(0)$
37. $f^{(n)}(t)$	$s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$		

Laplace transform properties

Property name	Time domain function	Laplace transform	Comment
	$f(t)$	$F(s)$	
Linearity	$af(t)+bg(t)$	$aF(s) + bG(s)$	a, b are constant
Scale change	$f(at)$	$\frac{1}{a}F\left(\frac{s}{a}\right)$	$a > 0$
Shift	$e^{-at}f(t)$	$F(s+a)$	
Delay	$f(t-a)$	$e^{-as}F(s)$	
Derivation	$\frac{df(t)}{dt}$	$sF(s) - f(0)$	
N-th derivation	$\frac{d^n f(t)}{dt^n}$	$s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \dots - f^{(n-1)}(0)$	
Power	$t^n f(t)$	$(-1)^n \frac{d^n F(s)}{ds^n}$	
Integration	$\int_0^t f(x)dx$	$\frac{1}{s}F(s)$	
Reciprocal	$\frac{1}{t}f(t)$	$\int_s^\infty F(x)dx$	
Convolution	$f(t) * g(t)$	$F(s) \cdot G(s)$	* is the convolution operator
Periodic function	$f(t) = f(t+T)$	$\frac{1}{1 - e^{-sT}} \int_0^T e^{-sx} f(x)dx$	

4.2 Properties of Laplace transform

Linearity

If $x_1(t) \leftrightarrow X_1(s)$, and $x_2(t) \leftrightarrow X_2(s)$
Then $ax_1(t) + bx_2(t) \leftrightarrow aX_1(s) + bX_2(s)$

Ex. Find Laplace Transform of $\sin\omega_0 t$

$$\sin\omega_0 t = \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}$$

$$L\{\sin\omega_0 t\} = \frac{1}{2j} \{L(e^{j\omega_0 t}) - L(e^{-j\omega_0 t})\} = \frac{1}{2j} \left[\frac{1}{s-j\omega_0} - \frac{1}{s+j\omega_0} \right]$$

$$L\{\sin\omega_0 t\} = \frac{\omega_0}{s^2 + \omega_0^2}$$

4.2 Properties of Laplace transform

Time-Shifting

If $x(t) \leftrightarrow X(s)$, Then $x(t - t_0)u(t - t_0) \leftrightarrow e^{-t_0 s} X(s)$

Ex. $x(t) = \text{rect}\left(\frac{t-a}{2a}\right)$, Find $X(s)$

$$x(t) = \text{rect}\left(\frac{t-a}{2a}\right) = u(t) - u(t - 2a)$$

$$X(s) = \frac{1}{s} - e^{-2as} \frac{1}{s}$$

4.2 Properties of Laplace transform

Shifting in the S Domain

If $x(t) \leftrightarrow X(s)$, Then $e^{s_0 t} x(t) \leftrightarrow X(s - s_0)$

Ex. Find $L[e^{-at} \sin \omega_0 t]$

From Laplace Table we have $L[\sin \omega_0 t] = \frac{\omega_0}{s^2 + \omega_0^2}$

$$L[e^{-at} \sin \omega_0 t] = \frac{\omega_0}{(s + a)^2 + \omega_0^2}$$

4.2 Properties of Laplace transform

Time Scaling

$$\text{If } x(t) \leftrightarrow X(s), \quad \text{Then } x(\alpha t) \leftrightarrow \frac{1}{\alpha} X\left(\frac{s}{\alpha}\right)$$

Ex Find $L\{u(\alpha t)\}$, $L\{u(\alpha t)\} = (1/\alpha) 1/s/\alpha = 1/s$

The result is expected, since $u(\alpha t) = u(t)$ for $\alpha > 0$

Differentiation & Integration in the Time Domain

$$\text{If } x(t) \leftrightarrow X(s), \quad \text{Then } \frac{dx(t)}{dt} \leftrightarrow sX(s) - X(0)$$

$$\frac{d^n x(t)}{dt^n} \leftrightarrow s^n X(s) - s^{n-1} X(0) - s^{n-2} X'(0) \dots \dots - X^{n-1}(0)$$

$$\int_0^t x(\tau) d\tau \leftrightarrow \frac{1}{s} X(s)$$

4.2 Properties of Laplace transform

Ex. $y''(t) + 3y'(t) + 2y(t) = 0$ $y(0) = 3, y'(0) = 1$

Find $y(t)$

Solution

$$[s^2Y(s) - sy(0) - y'(0)] + 3[sY(s) - y(0)] + 2Y(s) = 0$$

$$s^2Y(s) - 3s - 1 + 3sY(s) - 9 + 2Y(s) = 0$$

$$s^2 + 3s + 2]Y(s) = 3s + 10$$

$$\therefore Y(s) = \frac{3s + 10}{s^2 + 3s + 2} \quad \rightarrow y(t) = L^{-1}\{Y(s)\}$$

4.2 Properties of Laplace transform

Differentiation in The S-Domain

$$\text{If } x(t) \leftrightarrow X(s),$$
$$\text{Then } -tx(t) \leftrightarrow \frac{dX(s)}{ds}, \quad (t)^n x(t) \leftrightarrow (-1)^n \frac{d^n X(s)}{ds^n}$$

Ex Given $r(t)=t u(t)$, Find $R(s)$

Solution $R(s) = -\frac{dX(s)}{ds} L\{u(t)\} = -\frac{dX(s)}{ds} u(s) = -\frac{dX(s)}{ds} \left(\frac{1}{s}\right) = \frac{1}{s^2}$

Ex. Find $L[t^2 e^{-at}]$

Solution $L[t^2 e^{-at}] = -\frac{d^2}{ds^2} \left[\frac{1}{s+a} \right] = \frac{2}{(s+a)^3}$

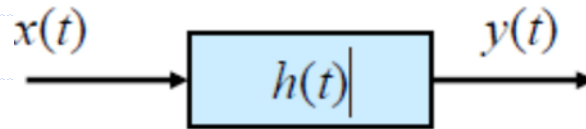
4.2 Properties of Laplace transform

Convolution

If $x(t) \leftrightarrow X(s)$, and $h(t) \leftrightarrow H(s)$
Then $x(t) * h(t) \leftrightarrow X(s)H(s)$

Ex given $x(t) = e^{-2t}u(t)$, $y(t) = \frac{2}{3}(e^{-t} + e^{-2t} - e^{-3t})u(t)$

Find $h(t)$



Solution $Y(s) = X(s)H(s) \Rightarrow H(s) = Y(s)/X(s)$

$$X(s) = \frac{1}{s+2}, \quad Y(s) = \frac{2/3}{s+1} + \frac{2/3}{s+2} - \frac{2/3}{s+3}$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{2(s^2 + 6s + 7)}{3(s+1)(s+3)}, \rightarrow h(t) = L^{-1}\{H(s)\}$$

4.2 Properties of Laplace transform

Initial-Value Theorem

$$x(0) = \lim_{s \rightarrow \infty} sX(s)$$

This property is useful, since it allows us to compute the initial value of the signal $x(t)$ directly from the Laplace transform $X(s)$ without having to find the inverse $x(t)$

Ex Given $X(s) = \frac{-3s^2+2}{s^3+s^2+3s+2}$ Find $x(0)$

$$\begin{aligned} x(0) &= \lim_{s \rightarrow \infty} sX(s) = \lim_{s \rightarrow \infty} \frac{-3s^3 + 2s}{s^3 + s^2 + 3s + 2} \\ &= \lim_{s \rightarrow \infty} \frac{-3 + 2/s^2}{1 + 1/s + 3/s^2 + 2/s^3} = -3 \end{aligned}$$

4.2 Properties of Laplace transform

Final-Value Theorem

$$\lim_{t \rightarrow \infty} x(t) = \lim_{s \rightarrow 0} sX(s)$$

- Final-value Theorem exists only if the system is stable
- Final-value Theorem is useful in many applications such as control theory, where we may need to find the final value (steady-state value) of the output of the system without solving for time domain

4.2 Properties of Laplace transform

Ex. Given $F(s) = \frac{5}{s(s^2+s+2)}$, Find $F(\infty)$

Solution

$$\lim_{s \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} sF(s) = \lim_{s \rightarrow 0} s \frac{5}{s(s^2 + s + 2)} = \frac{5}{2}$$

Ex. Given $F(s) = \frac{3}{s(s-2)}$, Find $F(\infty)$

Solution System is unstable so there is no final value

Stability

Stability conditions for an LTIC system

- Asymptotically stable if and only if all the poles of $H(s)$ are in left-hand plane (LHP). The poles may be repeated or non-repeated.
 - Unstable if and only if either one or both of these conditions hold
 - (i) at least one pole of $H(s)$ is in right-hand plane (RHP)
 - (ii) repeated poles of $H(s)$ are on the imaginary axis
 - A system is said to be “marginally stable” if it has at least one distinct pole on the $j\omega$ axis but no repeated poles on $j\omega$ axis
- Marginally breaks

Stability

- In most applications we desire a stable system
- We can easily check for stability by looking to see where the system's poles are

Example i. $H(s) = \frac{s+1}{(s+2)(s+3)}$

ii. $H(s) = \frac{s+1}{(s+2)(s-3)}$

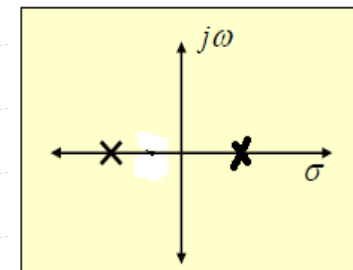
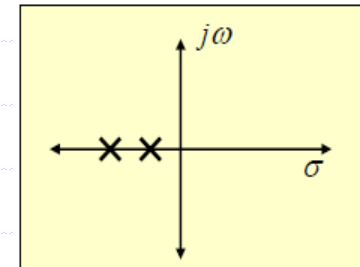
Solution

i. All poles are on LHP system is stable

$$h(t) = k_1 e^{-2t} + k_2 e^{-3t}$$

ii. One pole on RHP system is unstable

$$h(t) = k_1 e^{-2t} + k_2 e^{3t}$$



Inverse Laplace Transform

The function $X(s)$ has to be a proper rational function to find the inverse of Laplace transform.

The basic procedure is to express $X(s)$ as a summation of terms whose inverse Laplace transform are available in a table.

There are four general forms of solving the partial fraction; the roots of $D(s)$ are either:

1. Real and Distinct
2. Complex and Distinct
3. Real and Repeated
4. Complex and Repeated

Inverse Laplace Transform

Real simple Poles

Ex. $X(s) = \frac{2s^3+2s-10}{s^3+3s^2-s-3}$ Find $x(t)$

Solution:

$$\frac{2s^3+2s-10}{s^3+3s^2-s-3} = 2 + \frac{-6s^2+4s-4}{(s+1)(s-1)(s+3)}$$

$X_1(s)$

$$X_1(s) = \frac{-6s^2+4s-4}{(s+1)(s-1)(s+3)} = \frac{k_1}{s+1} + \frac{k_2}{s-1} + \frac{k_3}{s+3}$$

$$k_1 = (s+1)X_1(s)|_{s=-1} = \frac{-6s^2+4s-4}{(s-1)(s+3)}|_{s=-1} = \frac{-14}{-4} = 3.5$$

Inverse Laplace Transform

$$k_2 = (s - 1)X_1(s)|_{s=1} = \frac{-6s^2 + 4s - 4}{(s + 1)(s + 3)} \Big|_{s=1} = \frac{-6}{8} = -0.75$$

$$k_3 = (s + 3)X_1(s)|_{s=-3} = \frac{-6s^2 + 4s - 4}{(s + 1)(s - 1)} \Big|_{s=-3} = \frac{-70}{8} = -8.75$$

$$X(s) = 2 + \frac{3.5}{s+1} - \frac{0.75}{s-1} - \frac{8.75}{s+3}$$

$$x(t) = 2\delta(t) + [3.5e^{-t} - 0.75e^t - 8.75e^{-3t}]u(t)$$

$$x(0) = 2 - 6 = -4$$

$$X_1(0) = \lim_{s \rightarrow \infty} sX(s) = \lim_{s \rightarrow \infty} \frac{-6s^3 + 4s^2 - 4s}{(s+1)(s-1)(s+3)} = -6$$

$$x(0) = 2 - X_1(0) = 2 - 6 = -4$$

Inverse Laplace Transform

Repeated Real Poles

Ex. $X(s) = \frac{2s^2+3s-1}{(s+1)(s+2)^3}$ Find $x(t)$

Solution

$$X(s) = \frac{k_1}{s+1} + \frac{k_2}{(s+2)^3} + \frac{k_3}{(s+2)^2} + \frac{k_4}{s+2}$$

$$k_1 = \frac{2s^2+3s-1}{(s+2)^3} \Big|_{s=-1} = -2$$

$$k_2 = \frac{2s^2+3s-1}{s+1} \Big|_{s=-2} = -1$$

$$k_3 = \frac{d}{ds} \left(\frac{2s^2+3s-1}{s+1} \right) \Big|_{s=-2} = 4, \quad k_4 = \frac{1}{2!} \frac{d^2}{ds^2} \left(\frac{2s^2+3s-1}{s+1} \right) \Big|_{s=-2} = 2$$

Inverse Laplace Transform

$$X(s) = \frac{-2}{s+1} - \frac{1}{(s+2)^3} + \frac{4}{(s+2)^2} + \frac{2}{s+2}$$

$$x(t) = \left[-2e^{-t} - \frac{1}{2}t^2e^{-2t} + 4te^{-2t} + 2e^{-2t} \right] u(t)$$

Simple Complex Poles

Ex. $X(s) = \frac{100(s+3)}{(s+6)(s^2+6s+25)}$ find $x(t)$

Solution

$$X(s) = \frac{100(s+3)}{(s+6)(s^2+6s+25)} = \frac{100(s+3)}{(s+6)(s+3 \pm j4)}$$

Inverse Laplace Transform

$$X(s) = \frac{100(s+3)}{(s+6)(s+3 \pm j4)} = \frac{k_1}{s+6} + \frac{k_2}{s+3-j4} + \frac{k_2^*}{s+3+j4}$$

$$k_1 = (s+6)X(s)|_{s=-6} = \frac{100(s+3)}{(s^2+6s+25)}|_{s=-6} = -12$$

$$k_2 = (s+3-j4)X(s)|_{s=-3+j4} = \frac{100(s+3)}{(s+6)(s+3+j4)}|_{s=-3+j4} =$$

$$k_2 = 10 \angle -53.13^\circ \quad k_2^* = 10 \angle 53.13^\circ$$

$$\therefore X(s) = \frac{-12}{s+6} + \frac{10 \angle -53.13^\circ}{s+3-j4} + \frac{10 \angle 53.13^\circ}{s+3+j4}$$

Inverse Laplace Transform

X(s) contains distinct complex roots:

$$L^{-1} \left\{ \frac{k}{(s + \alpha - j\beta)^r} + \frac{k^*}{(s + \alpha + j\beta)^r} \right\} \\ = \left[\frac{2|k|t^{r-1}}{(r-1)!} e^{-\alpha t} \cos(\beta t + \theta) \right] u(t)$$

$$x(t) = [-12e^{-6t} + 20e^{-3t} \cos(4t - 53.13)]u(t)$$

$$x(0) = 0$$

$$x(0) = \lim_{s \rightarrow \infty} sX(s) = \lim_{s \rightarrow \infty} \frac{100(s^2 + 3s)}{(s + 6)(s^2 + 6s + 25)} = 0$$

Inverse Laplace Transform

Repeated Complex Poles

Ex. $X(s) = \frac{768}{(s^2+6s+24)^2}$ Find $x(t)$

Solution:

$$X(s) = \frac{768}{(s^2+6s+24)^2} = \frac{768}{(s+3-j4)^2(s+3+j4)^2}$$

$$\frac{768}{(s+3-j4)^2(s+3+j4)^2} = \frac{k_1}{(s+3-j4)^2} + \frac{k_2}{s+3-j4} + \frac{k_1^*}{(s+3+j4)^2} + \frac{k_2^*}{s+3+j4}$$

$$k_1 = \frac{768}{(s+3+j4)^2} \Big|_{s=-3+j4} = \frac{768}{(j8)^2} = -12 \angle 0^\circ$$

$$k_2 = \frac{d}{ds} \left[\frac{768}{(s+3+j4)^2} \right] \Big|_{s=-3+j4} = 3 \angle -90^\circ$$

Inverse Laplace Transform

$$k_1^* = -12 \angle 0^\circ$$

$$k_2^* = 3 \angle 90^\circ$$

$$x(s) = \left[\frac{-12}{(s+3-j4)^2} + \frac{-12}{(s+3+j4)^2} \right] + \left[\frac{3 \angle -90^\circ}{s+3-j4} + \frac{3 \angle 90^\circ}{s+3+j4} \right]$$

$$L^{-1} \left\{ \frac{k}{(s + \alpha - j\beta)^r} + \frac{k^*}{(s + \alpha + j\beta)^r} \right\} \\ = \left[\frac{2|k|t^{r-1}}{(r-1)!} e^{-\alpha t} \cos(\beta t + \theta) \right] u(t)$$

$$x(t) = [-24te^{-3t} \cos 4t + 6e^{-3t} \cos(4t - 90)] u(t)$$

4.3 Solution of Differential & Integro-Differential Equations

The Laplace transform of differential equation is an algebraic equation that can be readily solved for $Y(s)$. Next we take the inverse Laplace transform of $Y(s)$ to find the desired solution $y(t)$

4.3 Solution of Differential & Integro-Differential Equations

Example 4.10 P. 371 Solve the following second-order linear differential equation:

$$\frac{d^2 y}{dt^2} + 5 \frac{dy}{dt} + 6y(t) = \frac{dx}{dt} + x(t)$$

$y(0) = 2, \dot{y}(0) = 1$ and input $x(t) = e^{-4t}u(t)$.

Solution

Time Domain

$$\begin{aligned} &\frac{dy}{dt} \\ &\frac{d^2 y}{dt^2} \\ &x(t) = e^{-4t}u(t) \\ &\frac{dx}{dt} \end{aligned}$$

Laplace (Frequency) Domain

$$sY(s) - y(0^-) = sY(s) - 2$$

$$s^2Y(s) - sy(0^-) - \dot{y}(0^-) = s^2Y(s) - 2s - 1$$

$$X(s) = \frac{1}{s+4}$$

$$sX(s) - x(0^-) = \frac{s}{s+4} - 0 = \frac{s}{s+4}$$

4.3 Solution of Differential & Integro-Differential Equations

$$[s^2Y(s) - 2s - 1] + 5[sY(s) - 2] + 6Y(s) = \frac{s}{s+4} + \frac{1}{s+4}$$

$$(s^2 + 5s + 6)Y(s) - (2s + 1) = \frac{s+1}{s+4}$$

$$Y(s) = \frac{2s^2 + 20s + 45}{(s^2 + 5s + 6)(s + 4)}$$

$$Y(s) = \frac{13/2}{s+2} - \frac{3}{s+3} - \frac{3/2}{s+4}$$

$$y(t) = \left(\frac{13}{2} e^{-2t} - 3e^{-3t} - \frac{3}{2} e^{-4t} \right) u(t)$$

4.3 Solution of Differential & Integro-Differential Equations

Zero-input & Zero-state Responses

The Laplace transform method gives the total response, which include zero-input and zero state components. It is possible to separate the two components if we so desire.

Let's think about where the terms come from:

$$\frac{d^2 y}{dt^2} + 5 \frac{dy}{dt} + 6y(t) = \frac{dx}{dt} + x(t)$$
$$(s^2 + 5s + 6)Y(s) - \underbrace{(2s + 11)}_{\text{Initial condition term}} = \frac{s + 1}{s + 4} \leftarrow \text{Input term}$$

4.3 Solution of Differential & Integro-Differential Equations

$$Y(s) = \underbrace{\frac{2s + 11}{s^2 + 5s + 6}}_{\text{zero-input component}} + \underbrace{\frac{s + 1}{(s + 4)(s^2 + 5s + 6)}}_{\text{zero-state component}}$$

$$= \left[\frac{7}{s + 2} - \frac{5}{s + 3} \right] + \left[\frac{-1/2}{s + 2} + \frac{2}{s + 3} - \frac{3/2}{s + 4} \right]$$

$$y(t) = \underbrace{(7e^{-2t} - 5e^{-3t}) u(t)}_{\text{zero-input response}} + \underbrace{\left(-\frac{1}{2}e^{-2t} + 2e^{-3t} - \frac{3}{2}e^{-4t}\right) u(t)}_{\text{zero-state response}}$$

Final Exam 2018

Consider the LTI system;

$$\ddot{y}(t) + 5\dot{y}(t) + 6y(t) = 2\dot{x}(t) + 2x(t)$$

$$y(0) = 1, \dot{y}(0) = 2 \quad \text{and for } t > 0, x(t) = e^{-t}u(t)$$

Determine the zero-input and zero state response of above system.

Solution

$$\frac{d^n x(t)}{dt^n} \leftrightarrow s^n X(s) - s^{n-1}X(0) - s^{n-2}X'(0) \dots \dots - X^{n-1}(0)$$

Final Exam 2018

$$s^2 Y(s) - s y(0) - y'(0) + s [s Y(s) - y(0)] + 6 Y(s) = 2 [s X(s) - x(0)] + 2 X(s)$$

$$Y(s) [s^2 + 5s + 6] - 5 - 7 = [2s + 2] X(s)$$

$$j \boxed{X(s) = \frac{1}{s+1}}$$

$$Y(s) [s^2 + 5s + 6] = \overset{\text{Input}}{\frac{2s+2}{s+1}} + \overset{\text{I.C}}{(s+7)}$$

$$Y(s) = \overset{\text{Input}}{\frac{2}{s^2+5s+6}} + \overset{\text{I.C}}{\frac{(s+7)}{s^2+5s+6}} = \left(\overset{\text{Input}}{\frac{k_1}{s+2}} + \overset{\text{I.C}}{\frac{k_2}{s+3}} \right) + \left(\overset{\text{Input}}{\frac{k_3}{s+2}} + \overset{\text{I.C}}{\frac{k_4}{s+3}} \right)$$

$$k_1 = 2, k_2 = -2, k_3 = 5, k_4 = -4$$

$$\therefore y(t) = \underset{\text{Input}}{(2e^{-2t} - 2e^{-3t})u(t)} + \underset{\text{I.C}}{(5e^{-2t} - 4e^{-3t})u(t)}$$

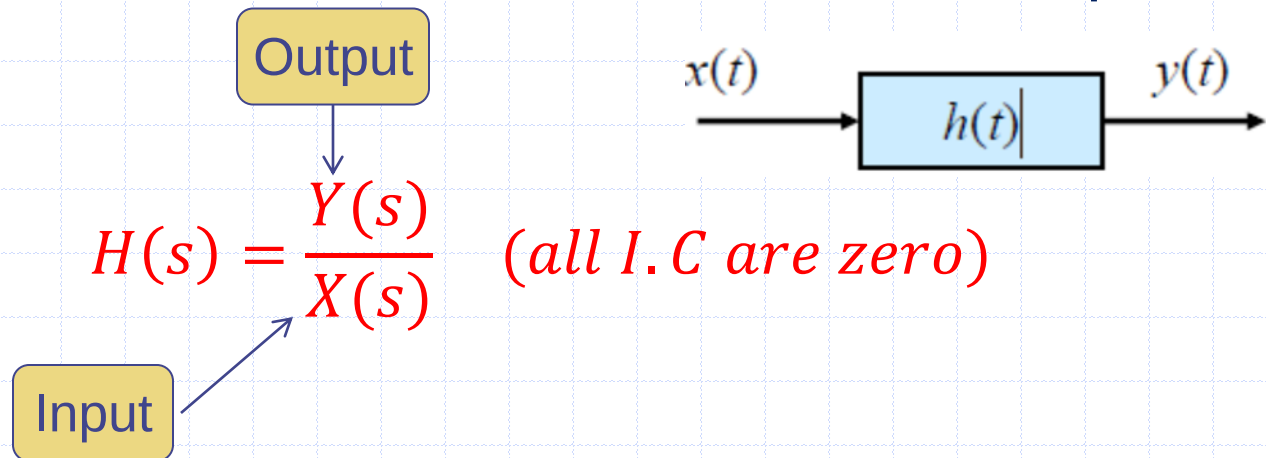
4.4 Analysis of Electrical Networks

How to compute T.F for circuit one way to find the T.F of the circuit is to compute its differential equation and then take its Laplace transform

However, it is generally simpler to compute T.F directly.

Transfer Function:

T.F is defined as the s-domain ratio of the output to the input



4.4 Analysis of Electrical Networks

We've seen that the system output's LT is:

$$Y(s) = \underbrace{\frac{C(s)}{A(s)}}_{\text{Part due to IC's}} + \underbrace{\frac{B(s)}{A(s)}}_{\text{Part due to Input}} X(s)$$

So, if the system is in zero-state then we only get the second term:

$$Y(s) = \frac{B(s)}{A(s)} X(s)$$

$\Downarrow$

$$Y(s) = H(s)X(s)$$

Define: $H(s) \triangleq \frac{B(s)}{A(s)}$

$H(s)$ = "transfer function"

$\Rightarrow$ System effect in zero-state case is completely set by the transfer function

4.4 Analysis of Electrical Networks

Poles and Zeros of a system

Given a system with Transfer Function:

$$H(s) = \frac{b_M s^M + b_{M-1} s^{M-1} + \dots + b_1 s + b_0}{s^N + a_{N-1} s^{N-1} + \dots + a_1 s + a_0}$$

$\leftarrow B(s)$
 $\leftarrow A(s)$

We can factor $B(s)$ and $A(s)$: (Recall: $A(s)$ = characteristic polynomial)

$$H(s) = \frac{b_M (s - z_1)(s - z_2) \dots (s - z_M)}{(s - p_1)(s - p_2) \dots (s - p_N)}$$

Pole-Zero Plot

This gives us a graphical view of the system's behavior

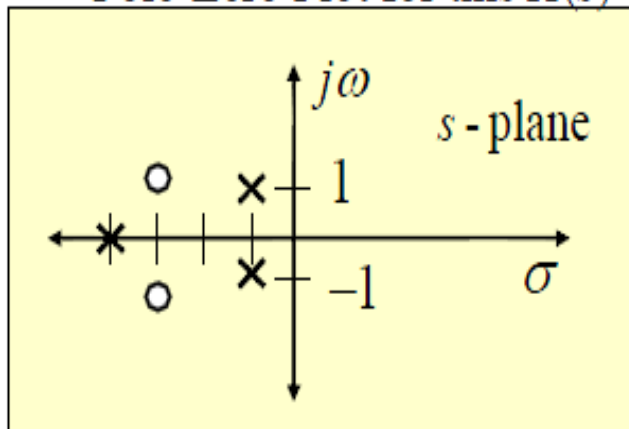
4.4 Analysis of Electrical Networks

Example

$$H(s) = \frac{2s^2 + 12s + 20}{s^3 + 6s^2 + 10s + 8} = \frac{2(s + 3 - j)(s + 3 + j)}{(s + 4)(s + 1 - j)(s + 1 + j)}$$

Real coefficients $\Rightarrow$ complex conjugate pairs

Pole-Zero Plot for this $H(s)$



Quiz

$$X(s) = \frac{e^{-2s}}{(s+1)^2(s+2)}$$

Find $x(t)$

Hint: If $x(t) \leftrightarrow X(s)$, Then $x(t - t_0)u(t - t_0) \leftrightarrow e^{-t_0 s} X(s)$

Quiz Solution

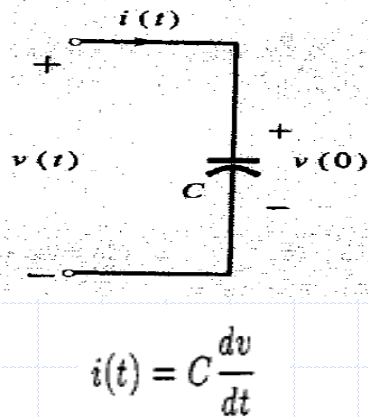
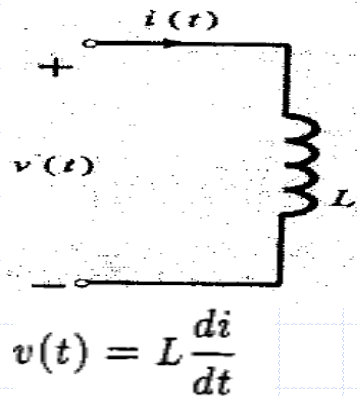
$$\textcircled{*} \quad \bar{X}(s) = \frac{e^{-2s}}{(s+1)^2 (s+2)} = e^{-2s} \cdot \frac{1}{(s+1)^2 (s+2)}$$

$\rightarrow \mathcal{L}^{-1} \left\{ (-e^{-t} + t e^{-t} + e^{-2t}) u(t) \right\}$

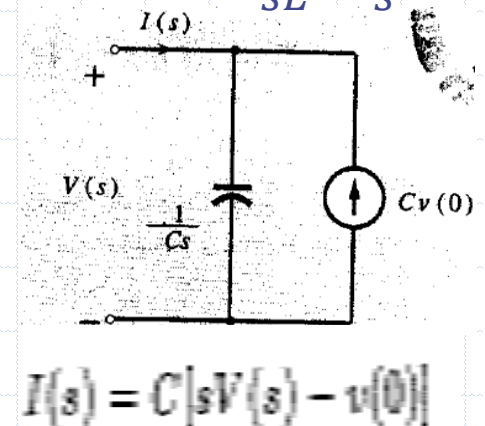
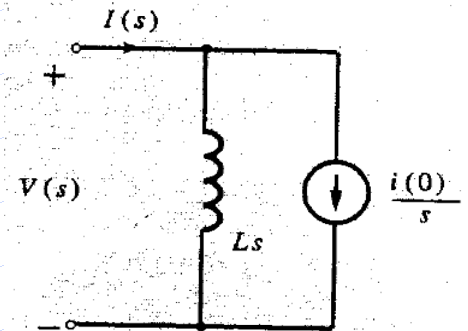
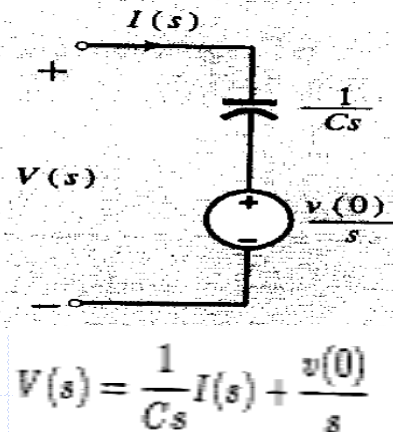
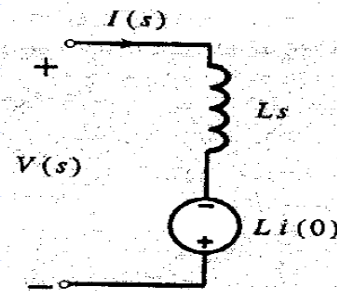
$$x(t) = \left[-e^{-(t-2)} + (t-2) e^{-(t-2)} + e^{-2(t-2)} \right] u(t-2)$$

4.4 Analysis of Electrical Networks

Time Domain



S- Domain



4.4 Analysis of Electrical Networks

Example: given the Circuit shown ,
find $y(t)$

Solution:

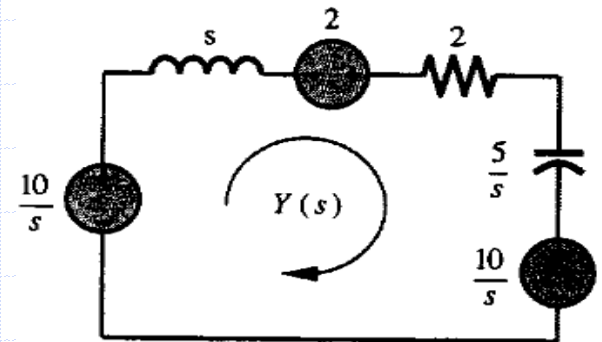
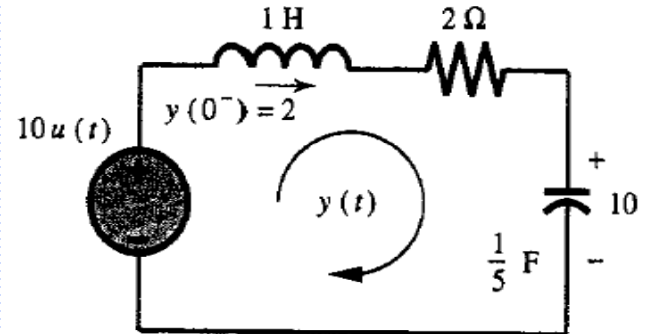
Apply Laplace Transform

The total voltage in the loop is

$$\frac{10}{s} + (2 + s + \frac{5}{s})Y(s) - \frac{10}{s} = 2$$

$$Y(s) = \frac{2}{s + 2 + \frac{5}{s}} = \frac{2s}{s^2 + 2s + 5}$$

$$y(t) = \sqrt{5}e^{-t} \cos[2t + 26.6^\circ] u(t)$$



4.4 Analysis of Electrical Networks

Exercise 4.4-1P 482

Find the zero state response $v_o(t)$, if the input voltage is $v(t) = te^{-t}u(t)$. Find TF, write differential eq relating $v_o(t)$ to $x(t)$.

Solution Loop Eq;

$$(1 + \frac{1}{s})Y_1(s) - \frac{1}{s}Y_2(s) = \frac{1}{(s+1)^2}$$

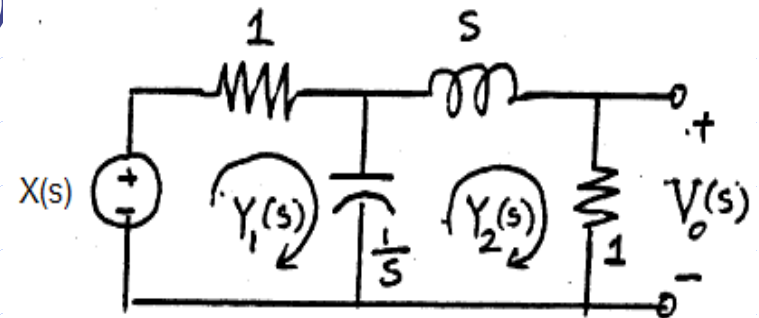
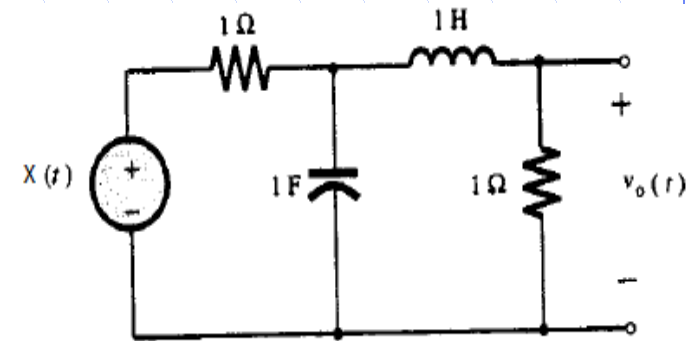
$$-\frac{1}{s}Y_1(s) + (s+1+\frac{1}{s})Y_2(s) = 0$$

$$\begin{bmatrix} \frac{s+1}{s} & -\frac{1}{s} \\ -\frac{1}{s} & \frac{s^2+s+1}{s} \end{bmatrix} \begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \begin{bmatrix} \frac{1}{(s+1)^2} \\ 0 \end{bmatrix}$$

Cramer rule yields

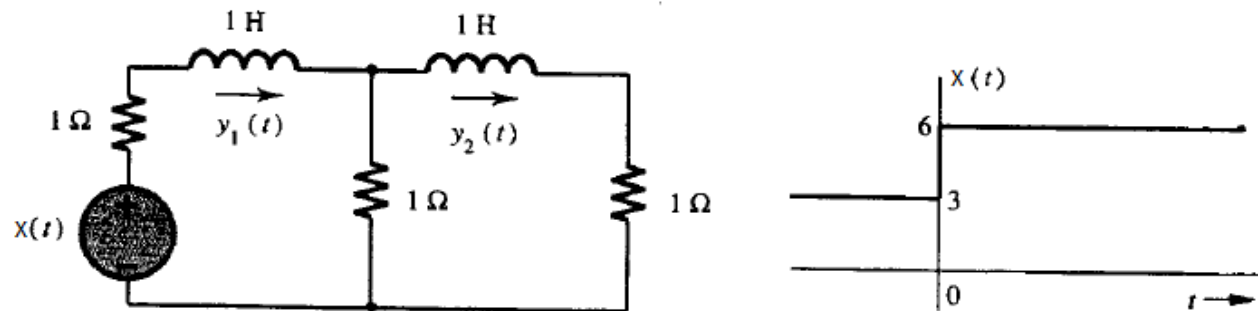
$$Y_2(s) = \frac{1}{(s+1)^2(s^2+2s+2)} = \frac{1}{(s+1)^2} - \frac{1}{s^2+2s+2}$$

$$v_o(t) = y_2(t) = (te^{-t} - \frac{1}{2}e^{-t}\sin t)u(t)$$



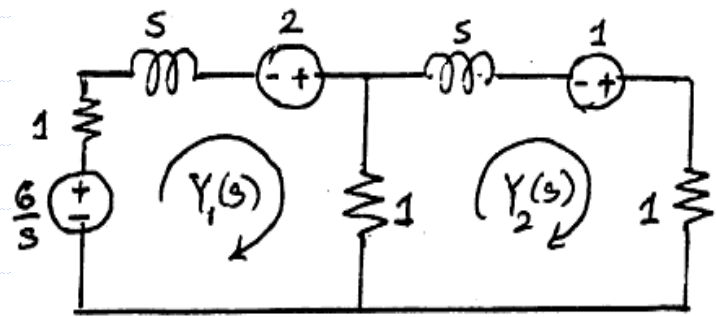
4.4 Analysis of Electrical Networks

Exercise 4.4-4 P 482 Find the loop currents $y_1(t), y_2(t)$,
Given $y_1(0) = 2, y_2(0) = 1$ for $t \geq 0$ for the input $x(t)$ as
 shown in Figure below



Solution: The Loop Eq. are

$$\begin{aligned}(s + 2)Y_1(s) - Y_2(s) &= 2 + \frac{6}{s} \\ -Y_1(s) + (s + 2)Y_2(s) &= 1\end{aligned}$$



4.4 Analysis of Electrical Networks

Cramer's rule yields

$$Y_1(s) = \frac{2s^2 + 11s + 12}{s(s+1)(s+3)} = \frac{4}{s} - \frac{3/2}{s+1} - \frac{1/2}{s+3}$$

$$Y_2(s) = \frac{s^2 + 4s + 6}{s(s+1)(s+3)} = \frac{2}{s} - \frac{3/2}{s+1} + \frac{1/2}{s+3}$$

$$y_1(t) = \left(4 - \frac{3}{2}e^{-t} - \frac{1}{2}e^{-3t}\right)u(t)$$

$$y_2(t) = \left(2 - \frac{3}{2}e^{-t} + \frac{1}{2}e^{-3t}\right)u(t)$$

Continuous-Time System Relationships

Time Domain

$$\frac{d^N y(t)}{dt^N} + a_{N-1} \frac{d^{N-1} y(t)}{dt^{N-1}} + \dots + a_1 \frac{dy(t)}{dt} + a_0 y(t) =$$

$$b_M \frac{d^M x(t)}{dt^M} + b_{M-1} \frac{d^{M-1} x(t)}{dt^{M-1}} + \dots + b_1 \frac{dx(t)}{dt} + b_0 x(t)$$

Differential
Equation

Char.
Modes

Impulse
Response

$h(t)$

Inspection in Practice
LT in Theory

Laplace Transform

Fourier Transform

Freq Domain

$$H(s) = \frac{b_M s^M + b_{M-1} s^{M-1} + \dots + b_1 s + b_0}{s^N + a_{N-1} s^{N-1} + \dots + a_1 s + a_0}$$

Transfer
Function

Pole-Zero
Diagram

$$s = j\omega$$

Frequency
Response

$$H(\omega) = H(s)|_{s=j\omega}$$

This Chart provides a “Roadmap” to the CT System Relationships!!!

1.From the differential equation you can get:

a. Transfer function, then the impulse response, the pole-zero plot, and if allowable you can get the frequency response

2.From the impulse response you can get:

a. Transfer function, then the Diff. Eq., the pole-zero plot, and if allowable you can get the frequency response

3.From the Transfer Function you can get:

a. Diff. Eq., the impulse response, the pole-zero plot, and if allowable you can get the frequency response

4.From the Frequency Response you can get:

a. Transfer function, then the Diff. Eq., the pole-zero plot, and the impulse response

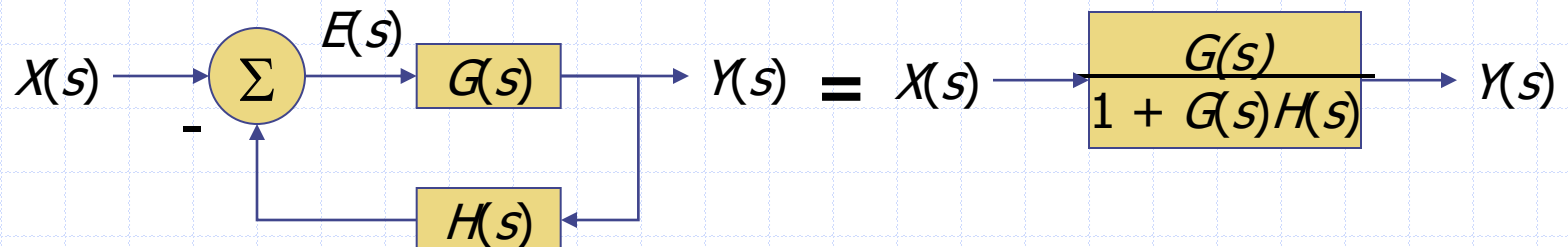
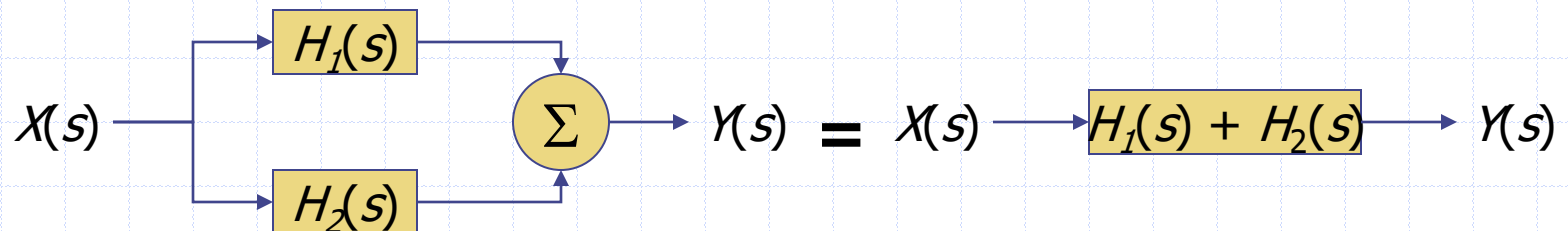
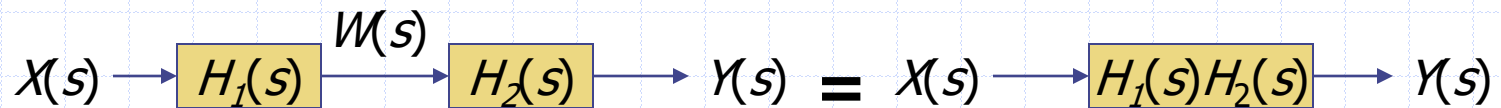
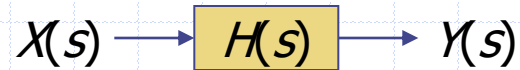
5.From the Pole-Zero Plot you can get:

a.(up to a scaling factor) Transfer function, then the Diff. Eq., the impulse response, and possible the Frequency Response

4.5 Block Diagrams

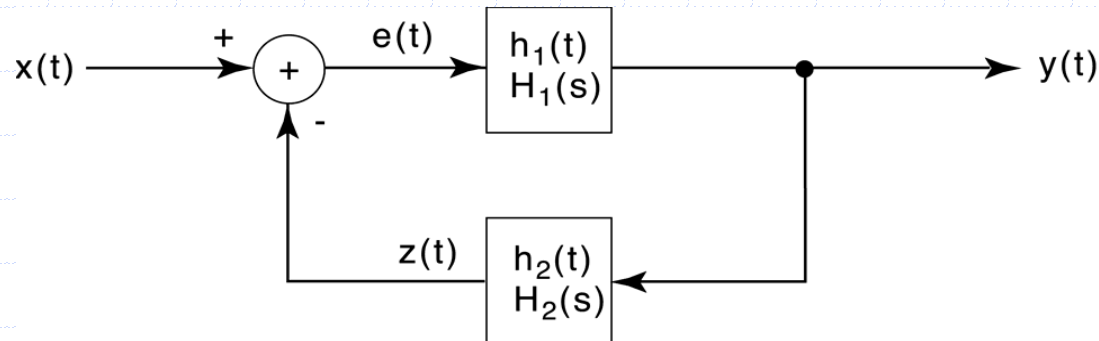
- Large systems may consist of an enormous number of components or elements. Analyzing such systems all at once could be next to impossible. In such cases, it is convenient to represent a system by suitably interconnected subsystems.
- Each subsystem can be characterized in terms of its input-output relationships.

4.5 Block Diagrams



4.5 Block Diagrams

Example: A basic feedback system consisting of block find TF



$$E(s) = X(s) - Z(s) = X(s) - H_2(s)Y(s)$$

$$Y(s) = H_1(s)E(s) = H_1(s)[X(s) - H_2(s)Y(s)]$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{H_1(s)}{1 + H_1(s)H_2(s)}$$

More on this later in
Control Course feedback