

Signal & Linear system

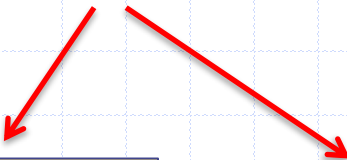
Chapter 5 DT System Analysis :
Z Transform

Basil Hamed

Introduction

Z-Transform does for DT systems what the Laplace Transform does for CT systems

Z-T is used to



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graph TD; A[Z-T is used to] --> B[Solve difference equations with initial conditions]; A --> C[Solve zero-state systems using the transfer function];
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Solve difference equations with initial conditions

Solve zero-state systems using the transfer function

In this chapter we will:

- Define the ZT
- See its properties
- Use the ZT and its properties to analyze D-T systems

Introduction

In mathematics and signal processing, the **Z-transform** converts a discrete-time signal, which is a sequence of real or complex numbers, into a **complex frequency-domain representation**.

It can be considered as a discrete-time equivalent of the Laplace transform.

- The basic idea now known as the Z-transform was known to Laplace, and it was re-introduced in 1947 by **W. Hurewicz**.
- It was later dubbed "the z-transform" by **Ragazzini** and **Zadeh** in the sampled-data control group at Columbia University in 1952

Introduction

What is the use of Z transform?

The z-transform is useful for the manipulation of discrete data sequences and has acquired a new significance in the formulation and analysis of discrete-time systems. It is used extensively today in the areas of applied mathematics, digital signal processing, control theory, population science, economics.

5.1 The Z-transform

We define $X(z)$, the direct Z-transform of $x[n]$, as

$$X[z] = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

Where z is the complex variable.

$$x[n] = \frac{1}{2\pi j} \oint X[z]z^{n-1}dz$$

The unilateral z-Transform

$$X[z] = \sum_{n=0}^{\infty} x[n]z^{-n}$$

Z-Transform of Elementary Functions:

Example 5.2 P 499 find the Z-transform of

a) $\delta[n]$

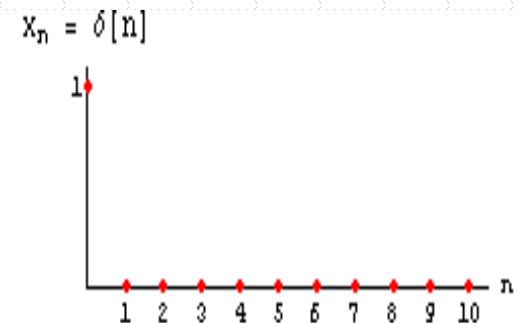
b) $U[n]$

c) $a^n u[n]$

d) $e^{-at} u(t)$

Solution

a) $x[n] = \delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$

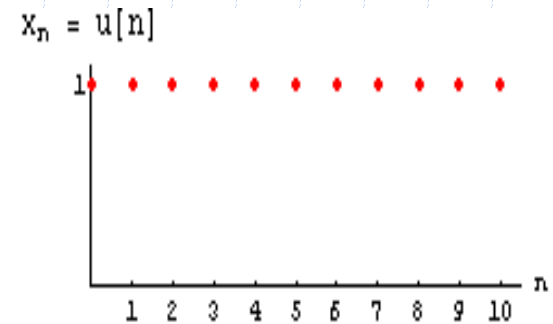


$$X[z] = \sum_{n=0}^{\infty} x[n]Z^{-n} = 1Z^0 + 0Z^{-1} + 0Z^{-2} + 0Z^{-3} + \dots = 1$$

$$\delta[n] \longleftrightarrow 1$$

Z-Transform of Elementary Functions:

$$b) x[n]=u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$



$$X[z] = \sum_{n=0}^{\infty} x[n]Z^{-n} = 1 + Z^{-1} + Z^{-2} + Z^{-3} + \dots$$

We have from power series from Book P 48

$$1 + x + x^2 + x^3 + x^3 + \dots = \frac{1}{1-x} \quad |x| < 1$$

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad |x| < 1$$

$$\therefore X[z] = \frac{1}{1-Z^{-1}} \quad |Z^{-1}| < 1$$



$$X[z] = \frac{z}{z-1} \quad |z| > 1$$

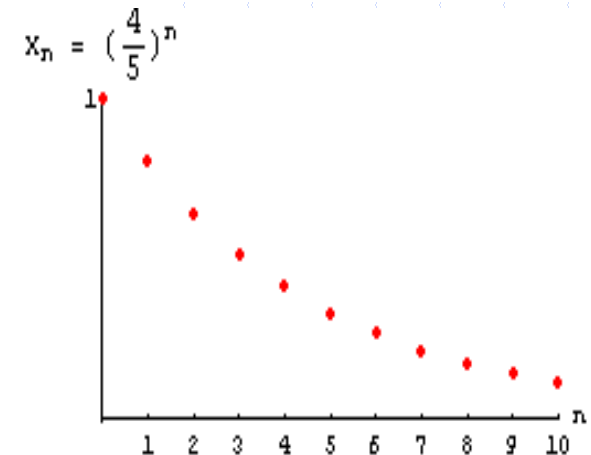
Z-Transform of Elementary Functions:

c) $x[n] = a^n u[n]$

$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n = \frac{z}{z-a} = \frac{1}{1-az^{-1}}$$

ROC = all z such that $|z| > |a|$

$$a^n u[n] \leftrightarrow \frac{z}{z-a} = \frac{1}{1-az^{-1}}$$



Z-Transform of Elementary Functions:

d) $x(t) = \begin{cases} e^{-at} & t \geq 0 \\ 0 & t < 0 \end{cases}$

$t \xrightarrow{\text{red arrow}} nT$

$$X[n] = e^{-anT}$$

$n = 0, 1, 2, \dots$

$$X[z] = Z[e^{-anT}] = \sum_{n=0}^{\infty} x[n] Z^{-n} = \sum_{n=0}^{\infty} e^{-anT} Z^{-n}$$

$$= \sum_{n=0}^{\infty} (e^{-aT} Z^{-1})^n = 1 + e^{-aT} Z^{-1} + e^{-2aT} Z^{-2} + \dots$$

$$X[z] = \frac{1}{1 - e^{-aT} Z^{-1}} = \frac{Z}{Z - e^{-aT}} \quad |Z| > e^{-aT}$$

Z-Transform of Elementary Functions:

Example given $x[n] = \begin{cases} \left[\frac{1}{2}\right]^n & n \geq 0 \\ 0 & n < 0 \end{cases}$

$$y[n] = \begin{cases} -\left[\frac{1}{2}\right]^n & n < 0 \\ 0 & n \geq 0 \end{cases}$$

Find $X(z)$ & $Y(z)$

Solution $X(z) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n Z^{-n} = \sum_{n=0}^{\infty} \left(\frac{1}{2} Z^{-1}\right)^n$

$$\therefore X(z) = \frac{1}{1 - \frac{1}{2} Z^{-1}} = \frac{Z}{Z - \frac{1}{2}} \quad |Z| > \frac{1}{2}$$

Region of Convergence

Example

$x[n] = a^n u[n]$ - right-sided

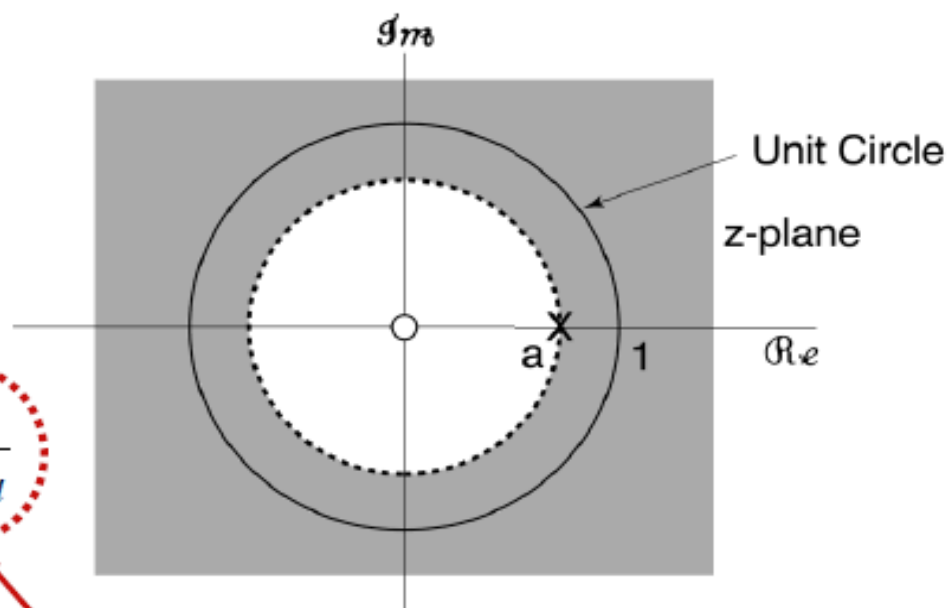
$$X(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n}$$

$$= \sum_{n=0}^{\infty} (az^{-1})^n$$

$$= \frac{1}{1 - az^{-1}} = \frac{z}{z - a}$$

If $|az^{-1}| < 1$, i.e., $|z| > |a|$

That is, ROC $|z| > |a|$,
outside a circle



This form to find
pole and zero locations

Region of Convergence

Example

$$x[n] = -a^n u[-n-1] \text{ - left-sided}$$

$$X(z) = \sum_{n=-\infty}^{\infty} \{-a^n u[-n-1] z^{-n}\}$$

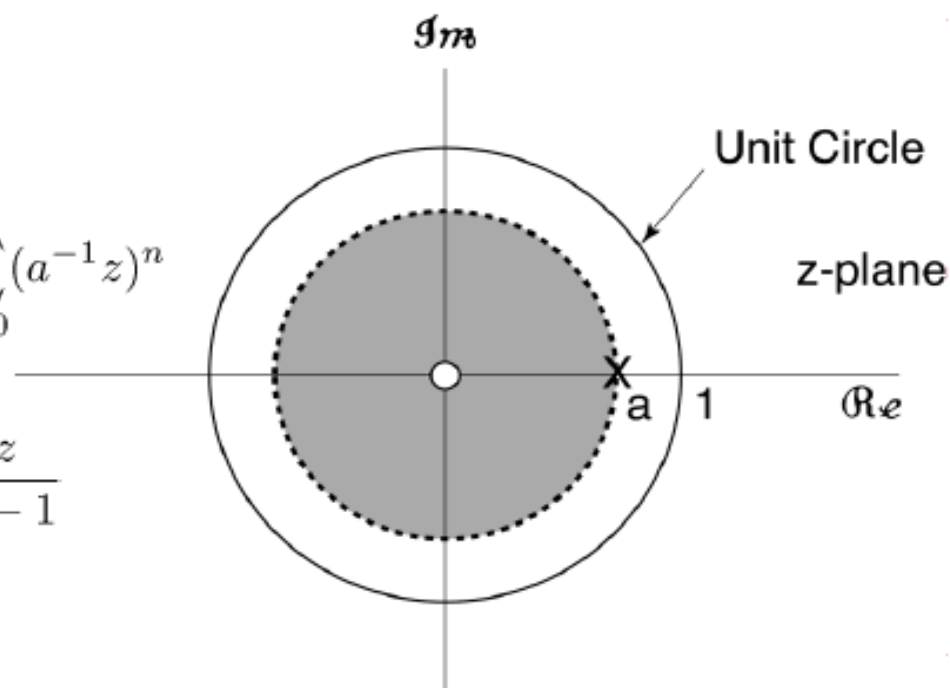
$$= - \sum_{n=-\infty}^{-1} a^n z^{-n}$$

$$= - \sum_{n=1}^{\infty} a^{-n} z^n = 1 - \sum_{n=0}^{\infty} (a^{-1} z)^n$$

$$= 1 - \frac{1}{1 - a^{-1} z} = \frac{a^{-1} z}{a^{-1} z - 1}$$

$$= \frac{z}{z - a},$$

If $|a^{-1} z| < 1$, i.e., $|z| < |a|$



Z-Transform of Elementary Functions:

$$Y(z) = \sum_{n=-\infty}^{-1} -\left(\frac{1}{2}\right)^n Z^{-n} = -\sum_{n=-\infty}^{-1} \left(\frac{1}{2} Z^{-1}\right)^n$$

Let $n=-m$

$$Y(z) = -\sum_{m=-\infty}^{-1} \left(\frac{1}{2} Z^{-1}\right)^{-m} = -\sum_{m=0}^{\infty} \left(\frac{1}{2Z}\right)^m$$

$$\begin{aligned} Y(z) &= \sum_{m=1}^{\infty} (2Z)^m = 1 - \sum_{m=0}^{\infty} (2Z)^m = 1 - \frac{1}{1-2Z} \quad |2Z| < 1 \end{aligned}$$

$$\therefore Y(z) = \frac{Z}{Z - \frac{1}{2}} \quad |Z| < \frac{1}{2}$$

As seen in the example above, $X(z)$ & $Y(z)$ are identical, the only different is ROC

Relationship between ZT & LT

$$x(t) \longleftrightarrow X(s) = \int_{-\infty}^{\infty} x(t)e^{-st} dt = \mathcal{L}\{x(t)\}$$

Let $t = nT$

$$\begin{aligned} &= \lim_{T \rightarrow 0} \sum_{n=-\infty}^{\infty} \underbrace{x(nT)}_{x[n]} (e^{sT})^{-n} \cdot T \\ &= \lim_{T \rightarrow 0} T \sum_{n=-\infty}^{\infty} x[n] (e^{sT})^{-n} \end{aligned}$$

The (Bilateral) z-Transform

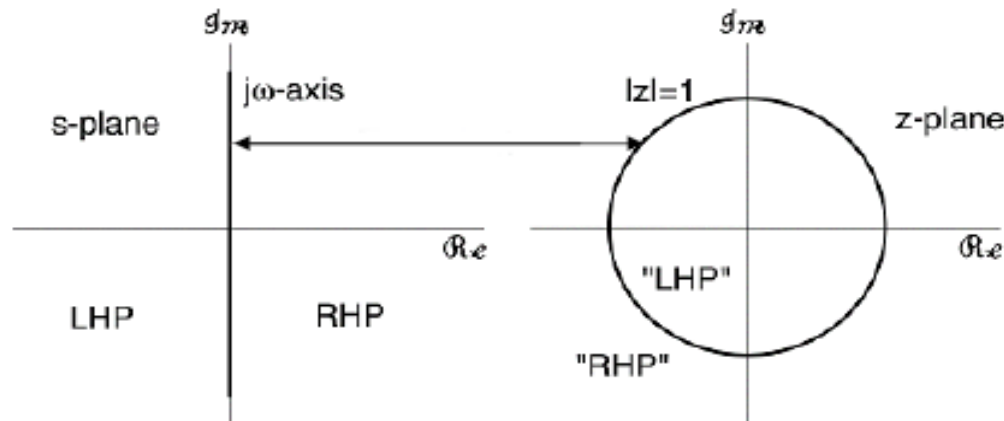
$$x[n] \longleftrightarrow X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \mathcal{Z}\{x[n]\}$$

Can think of z-transform as DT
version of Laplace transform with
 $z = e^{sT}$

Relationship between ZT & LT

$$e^{sT} = z$$

$j\omega$ axis in s -plane ($s = j\omega$) $\Leftrightarrow |z| = |e^{j\omega T}| = 1$ - a unit circle in z -plane



- LHP in s -plane, $Re(s) < 0 \Rightarrow |z| = |e^{sT}| < 1$, inside the $|z| = 1$ circle.
Special case, $Re(s) = -\infty \Leftrightarrow |z| = 0$.
- RHP in s -plane, $Re(s) > 0 \Rightarrow |z| = |e^{sT}| > 1$, outside the $|z| = 1$ circle.
Special case, $Re(s) = +\infty \Leftrightarrow |z| = \infty$.
- A vertical line in s -plane, $Re(s) = \text{constant} \Leftrightarrow |e^{sT}| = \text{constant}$, a circle in z -plane.

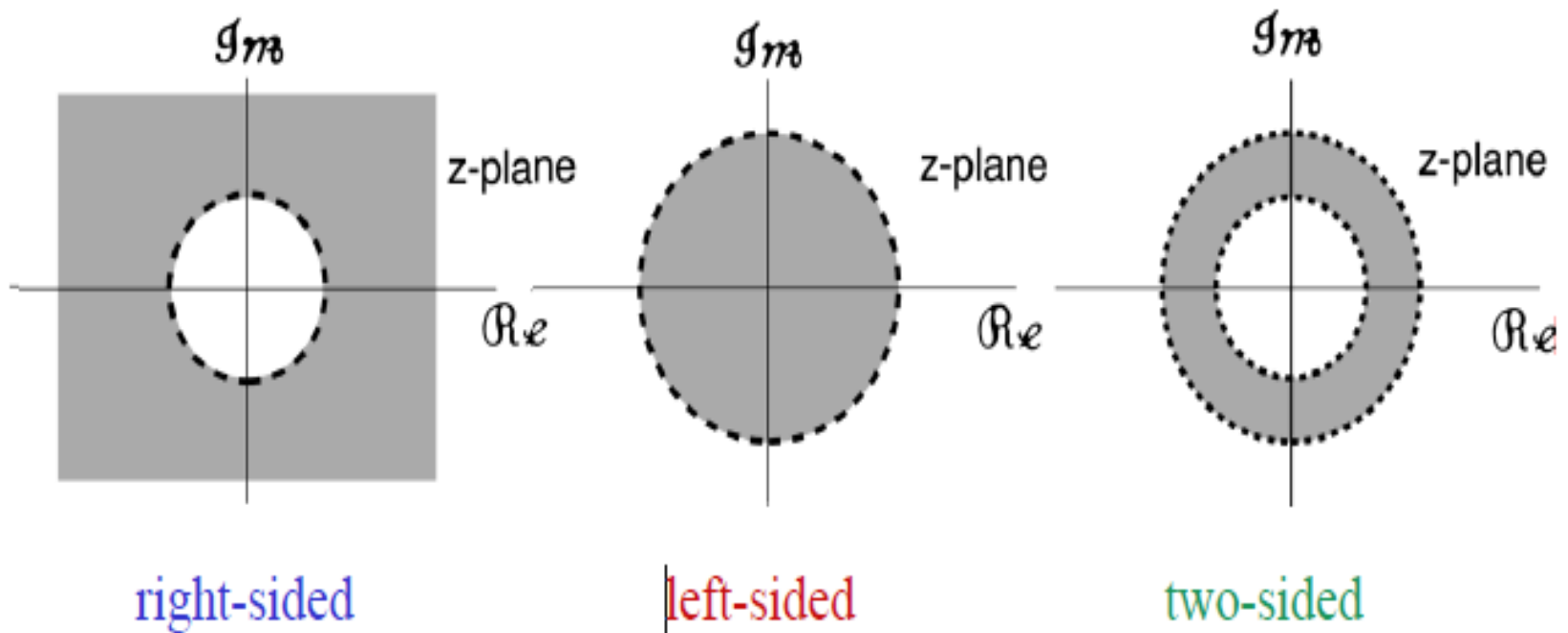
Relationship between ZT & LT

In the S-plane, the region of stability is the left half-plane. If the transfer function, $G(s)$, is transformed into a sampled-data transfer function, $G(z)$, the region of stability on the z-plane can be evaluated from the definition, $z = e^{Ts}$. Letting $s = \alpha + j\omega$, we obtain

$$\begin{aligned} z = e^{Ts} &= e^{T(\alpha + j\omega)} = e^{\alpha T} e^{j\omega T} \\ &= e^{\alpha T} (\cos \omega T + j \sin \omega T) \\ &= e^{\alpha T} \angle \omega T \end{aligned}$$

since $(\cos \omega T + j \sin \omega T) = 1 \angle \omega T$.

ROC



ROC

Example given $x[n] = (\frac{1}{2})^n u[n] + (\frac{1}{3})^n u[n]$ Find $X(z)$

Solution

$$X[z] = \sum_{n=0}^{\infty} (\frac{1}{2})^n Z^{-n} + \sum_{n=0}^{\infty} (\frac{1}{3})^n Z^{-n}$$

$$X[z] = \frac{1}{1 - \frac{1}{2}Z^{-1}} + \frac{1}{1 - \frac{1}{3}Z^{-1}} = \frac{Z}{Z - \frac{1}{2}} + \frac{Z}{Z - \frac{1}{3}}$$

$$\text{ROC} \quad |Z| > \frac{1}{2}, \quad |Z| > \frac{1}{3}$$

$$\therefore |Z| > \max\left(\frac{1}{2}, \frac{1}{3}\right) = \frac{1}{2}$$

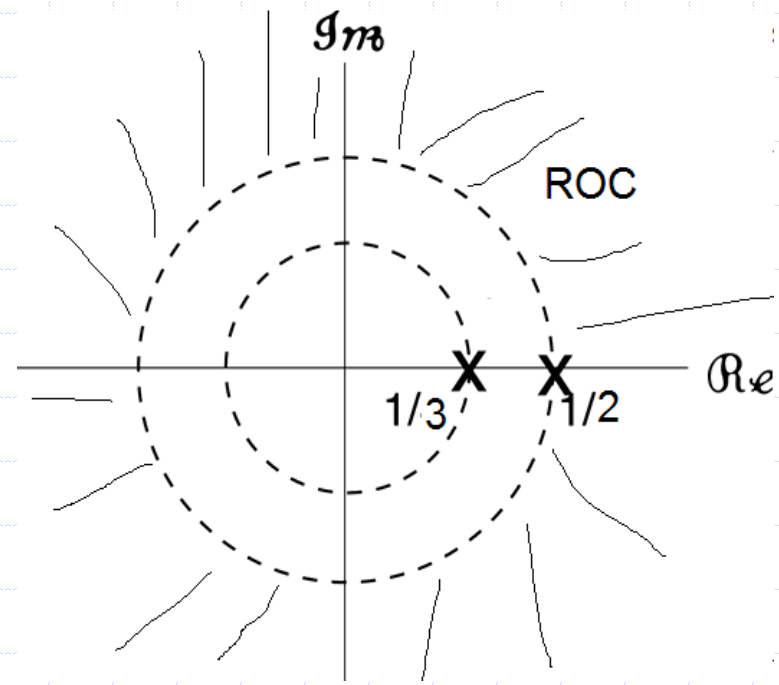


Table of z-transforms

	Sequence	z - transform
1	$\delta[n]$	1
2	$u[n]$	$\frac{z}{z-1}$
3	b^n	$\frac{z}{z-b}$
4	$b^{n-1}u[n-1]$	$\frac{1}{z-b}$
5	e^{an}	$\frac{z}{z-e^a}$
6	n	$\frac{z}{(z-1)^2}$
7	n^2	$\frac{z(z+1)}{(z-1)^3}$
8	$b^n n$	$\frac{bz}{(z-b)^2}$
9	$e^{an} n$	$\frac{ze^a}{(z-e^a)^2}$
10	$\sin(an)$	$\frac{\sin(a)z}{z^2-2\cos(a)z+1}$
11	$b^n \sin(an)$	$\frac{\sin(a)bz}{z^2-2\cos(a)bz+b^2}$
12	$\cos(an)$	$\frac{z(z-\cos(a))}{z^2-2\cos(a)z+1}$
13	$b^n \cos(an)$	$\frac{z(z-b\cos(a))}{z^2-2\cos(a)bz+b^2}$

Table 9.1. z-transforms of some common sequences.
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5.2 Some Properties of The Z-Transform

As seen in the Fourier & Laplace transform there are many properties of the Z-transform will be quite useful in system analysis and design.

Linearity

If $x_1[n] \leftrightarrow X_1[z]$ & $x_2[n] \leftrightarrow X_2[z]$

Then $ax_1[n] + bx_2[n] \leftrightarrow aX_1[z] + bX_2[z]$

Example Find z-transform of $x(n) = u(n) - (0.5)^n u(n)$.

Solution:

Using z- transform
table:

$$\left\{ \begin{array}{l} Z(u(n)) = \frac{z}{z-1} \\ Z(0.5^n u(n)) = \frac{z}{z-0.5} \end{array} \right.$$

$$X(z) = \frac{z}{z-1} - \frac{z}{z-0.5}$$

5.2 Some Properties of The Z-Transform

Time-Shifting

Right Shift of $x[n]$ (delay) if $x[n] \leftrightarrow X[z]$

Then $Z[x[n - n_0]] = Z^{-n_0}X[z] + X[-n_0] + Z^{-1}X[-n_0 + 1] + \dots$
 $\dots + Z^{-n_0+1}X[-1]$

Note that if $x[n]=0$ for $n=-1,-2,-3,\dots, -n_0$ then
 $Z\{x[n-n_0]\}=Z^{-n_0}X[z]$

$$\begin{aligned}x[n - 1] &\leftrightarrow Z^{-1}X[z] + x[-1] \\x[n - 2] &\leftrightarrow Z^{-2}X[z] + x[-2] + Z^{-1}x[-1]\end{aligned}$$

5.2 Some Properties of The Z-Transform

Left Shift in Time (Advanced)

$$x[n + 1] \leftrightarrow ZX[z] - x[0]Z$$

$$x[n + 2] \leftrightarrow Z^2X[z] - x[0]Z^2 - x[1]Z$$

$$\vdots$$

$$x[n + n_0] \leftrightarrow Z^{n_0}X[z] - x[0]Z^{n_0} - x[1]Z^{n_0-1} - \dots - x[n_0 - 1]Z$$

Example given $y[n] - \frac{1}{2}y[n - 1] = \delta[n]$, $y[-1] = 3$

Find $y[n]$

5.2 Some Properties of The Z-Transform

$$Y[z] - \frac{1}{2} [Z^{-1}Y[z] + y[-1]] = 1$$

$$Y[z] \left[1 - \frac{1}{2} Z^{-1} \right] = 1 + \frac{3}{2}$$

$$\therefore Y[z] = \frac{5/2}{1 - \frac{1}{2} Z^{-1}} = \frac{\frac{5}{2} Z}{Z - \frac{1}{2}}$$

$$y[n] = Z^{-1}\{Y[z]\}$$

5.2 Some Properties of The Z-Transform

Example Given $y[n+2] - y[n+1] + \frac{2}{9}y[n] = x[n]$

For $y[n]$, $n \geq 0$ if $x[n] = u[n]$, $y[1] = 1$, $y[0] = 1$

Solve the difference equation

Solution

$$[Z^2Y[z] - y[0]Z^2 - y[1]Z] - [ZY[z] - y[0]Z] + \frac{2}{9}Y[z] = X[z]$$

$$Y[z] \left[Z^2 - Z + \frac{2}{9} \right] - Z^2 - Z + Z = \frac{Z}{Z-1}$$

$$\therefore Y[z] = \frac{\frac{Z}{Z-1} + Z^2}{Z^2 - Z + \frac{2}{9}}$$

take inverse z and find $y[n]$

5.2 Some Properties of The Z-Transform

Example Find z-transform of $y(n) = (0.5)^{(n-5)} \cdot u(n-5)$,
where $u(n-5) = 1$ for $n \geq 5$ and $u(n-5) = 0$ for $n < 5$.

Solution:

Using shift theorem,

$$Y(z) = Z[(0.5)^{n-5} u(n-5)] = z^{-5} Z[(0.5)^n u(n)].$$

Using z-transform table

$$Y(z) = z^{-5} \cdot \frac{z}{z - 0.5} = \frac{z^{-4}}{z - 0.5}.$$

5.2 Some Properties of The Z-Transform

Frequency Scaling (Multiplication by a^n)

if $x[n] \leftrightarrow X[z]$ Then $a^n x[n] \leftrightarrow X\left[\frac{z}{a}\right]$

Example given $y[n] = (a^n \cos \Omega_0)u[n]$ Find $Y[z]$

Solution

From Z-Table

$$x[n] = (\cos \Omega_0)u[n] \rightarrow X[z] = \frac{Z(Z - \cos \Omega_0)}{Z^2 - 2Z \cos \Omega_0 + 1}$$

$$\therefore Y[z] = \frac{(Z/a)(Z/a - \cos \Omega_0)}{(Z/a)^2 - 2(Z/a) \cos \Omega_0 + 1}$$

$$Y[z] = \frac{Z(Z - a \cos \Omega_0)}{Z^2 - 2aZ \cos \Omega_0 + a^2}$$

5.2 Some Properties of The Z-Transform

Differentiation with Respect to Z

if $x[n] \leftrightarrow X[z]$ Then $Z \left[n^k x[n] \right] = \left(-Z \frac{d}{dz} \right)^k X(z)$

Example; given $y[n] = n[n+1]u[n]$, find $Y[z]$

Solution $y[n] = n^2 u[n] + nu[n]$

$$Z[n u[n]] = -Z \frac{d}{dz} Z[u[n]] = -Z \frac{d}{dz} \left[\frac{Z}{Z-1} \right] = \frac{Z}{(Z-1)^2}$$

$$\text{And } Z[n^2 u[n]] = \left(-Z \frac{d}{dz} \right)^2 Z[u[n]]$$

$$= -Z \frac{d}{dz} \frac{Z}{(Z-1)^2} = \frac{Z(Z+1)}{(Z-1)^3}$$

$$Y[z] = \frac{Z(Z+1)}{(Z-1)^3} + \frac{Z}{(Z-1)^2} = \frac{2Z^2}{(Z-1)^3}$$

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5.2 Some Properties of The Z-Transform

Initial Value Theorem

if $x[n] \leftrightarrow X[z]$ Then $x(0) = \lim_{Z \rightarrow \infty} X(z)$

Example $\frac{X(z)}{z} = \frac{Z^3 - \frac{3}{4}Z^2 + 2Z - \frac{5}{4}}{(Z-1)(Z-\frac{1}{3})(Z^2 - \frac{1}{2}Z + 1)}$ find $x(0)$

Solution

$$x(0) = \lim_{Z \rightarrow \infty} X(z)$$

$$x(0) = \lim_{Z \rightarrow \infty} \frac{Z^4 - \frac{3}{4}Z^3 + 2Z^2 - \frac{5}{4}Z}{Z^4 - \frac{11}{6}Z^3 + 2Z^2 - \frac{9}{6}Z + \frac{1}{3}} = 1$$

5.2 Some Properties of The Z-Transform

The initial value theorem is a convenient tool for checking if the Z-transform of a given signal is in error.

Using Matlab software we can have $x[n]$;

$$x[n] = u[n] + \left(\frac{1}{3}\right)^n u[n] - \left(\frac{1}{2}\right)^n \cos\left(\frac{\pi}{3} n\right)$$

The initial value is $x(0)=1$, which agrees with the result we have.

Final value Theorem

$$\lim_{n \rightarrow \infty} x[n] = \lim_{Z \rightarrow 1} (Z - 1)X(z) = \lim_{Z \rightarrow 1} (1 - Z^{-1})X(z)$$

5.2 Some Properties of The Z-Transform

As in the continuous-time case, care must be exercised in using the final value thm. For the existence of the limit; all poles of the system must be inside the unit circle. (system must be stable)

Example given $X(z) = \frac{z^3 - \frac{3}{4}z^2 + 2z - \frac{5}{4}}{(z-1)(z-\frac{1}{3})(z^2 - \frac{1}{2}z + 1)}$ Find $x(\infty)$

Solution $x(\infty) = \lim_{z \rightarrow 1} (1 - z^{-1})X(z) = \frac{z^3 - \frac{3}{4}z^2 + 2z - \frac{5}{4}}{z(z-\frac{1}{3})(z^2 - \frac{1}{2}z + 1)} = 1$

Example given $x[n] = 2^n u[n]$ Find $x(\infty)$

Solution $x(z) = \frac{z}{z-2}$ The system is unstable because we have one pole outside the unit circle so the system does not have final value,

5.2 Some Properties of The Z-Transform

Example Find the Initial and Final value of $x[n]$ whose signal is given by $X(Z) = 2 + 3Z^{-1} + 4Z^{-2}$

Solution the initial value of the signal by apply theorem
$$x(0) = \lim_{Z \rightarrow \infty} X(z) = \lim_{Z \rightarrow \infty} [2 + 3Z^{-1} + 4Z^{-2}]$$

$$= 2 + \left(\frac{3}{\infty}\right) + \left(\frac{4}{\infty}\right) = 2$$

Final value of signal applying the theorem

$$x(\infty) = \lim_{Z \rightarrow 1} (1 - Z^{-1})X(z)$$

$$= \lim_{Z \rightarrow 1} [(1 - Z^{-1})(2 + 3Z^{-1} + 4Z^{-2})]$$

$$= \lim_{Z \rightarrow 1} [2 + Z^{-1} + Z^{-2} - 4Z^{-3}] = 2 + 1 + 1 - 4 = 0$$

Stability of DT Systems

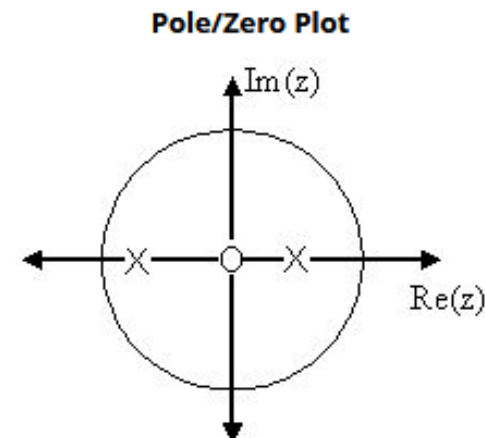
A discrete-time LTI system with impulse response h is stable if and only if the transfer function $\hat{H}$, which is the Z transform of h , has a region of convergence that includes the unit circle.

For the stability of the system function If the discrete rational system is stable then $\sum |h(x)| < \infty$

$$H(z) = \frac{z}{\left(z - \frac{1}{2}\right)\left(z + \frac{3}{4}\right)}$$

The zeros are: $\{0\}$ The poles are: $\left\{\frac{1}{2}, -\frac{3}{4}\right\}$

System is stable



Stability of DT Systems

Result:

$$\sum_{n=0}^{\infty} |h[n]| < \infty$$

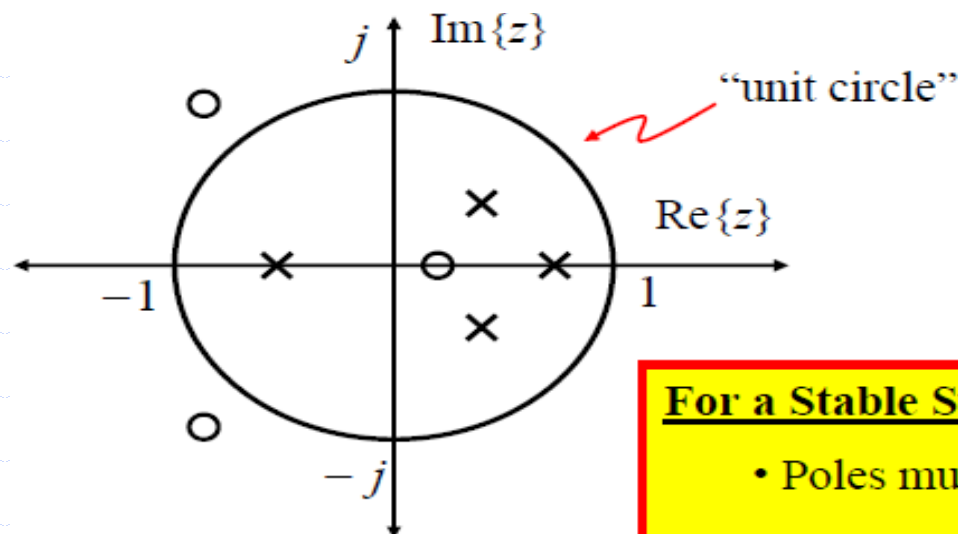
$\Downarrow$
System is stable

is equivalent to

$$\text{all } |p_i| < 1$$

$\Downarrow$
System is stable

i.e., poles are inside the unit circle



For a Stable System

- Poles must be "inside unit circle"
- Zeros can be anywhere

Stability of DT Systems

Example

$$H(z) = \frac{z}{\left(z - \frac{1}{2}\right)\left(z + \frac{3}{4}\right)}$$

Region of Convergence for the Pole/Zero Plot

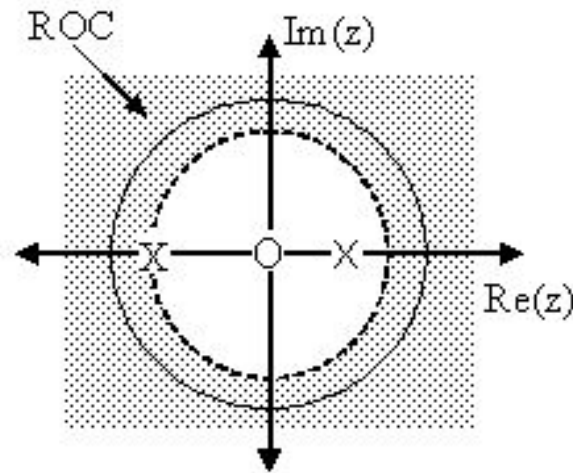
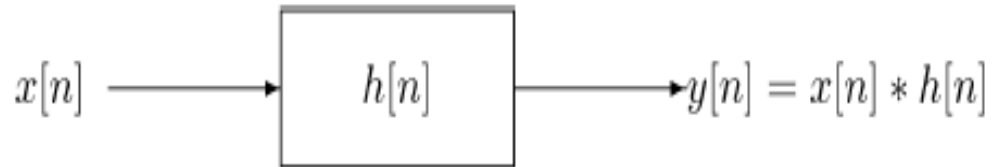


Figure : The shaded area represents the chosen ROC for the transfer function.

5.2 Some Properties of The Z-Transform

Convolution



$$Y(z) = X(z)H(z)$$

Example: given $h[n] = \{1, 2, 0, -1, 1\}$ and $x[n] = \{1, 3, -1, -2\}$

Find $y[n]$

Solution $y[n] = x[n] * h[n] \quad \longrightarrow \quad Y(z) = X(z)H(z)$

$$X(z) = 1 + 3Z^{-1} - Z^{-2} - 2Z^{-3}$$

$$H(z) = 1 + 2Z^{-1} + 0 - Z^{-3} + Z^{-4}$$

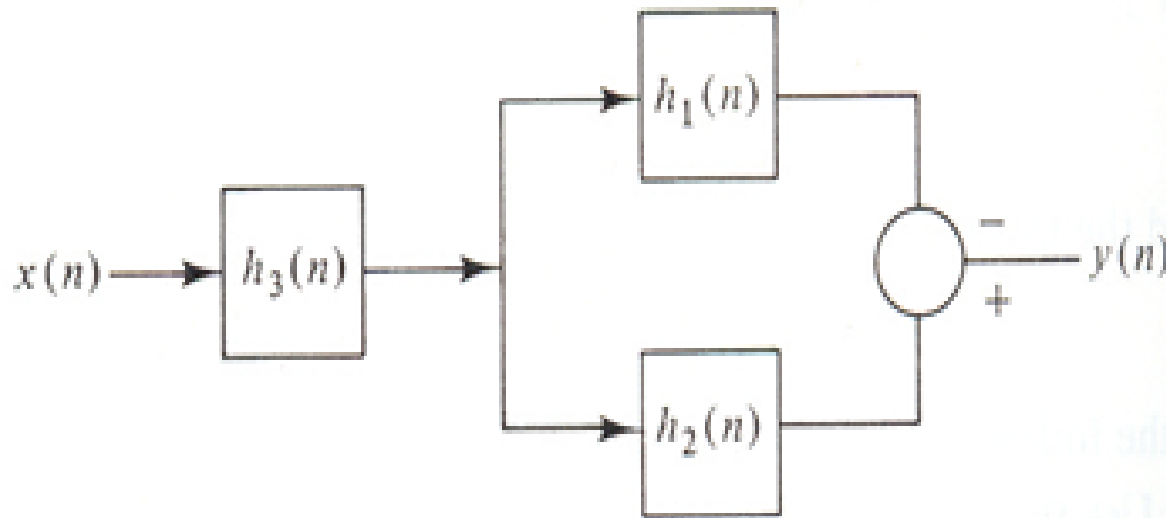
$$Y(z) = 1 + 5Z^{-1} + 5Z^{-2} - 5Z^{-3} - 6Z^{-4} + 4Z^{-5} + Z^{-6} - 2Z^{-7}$$

$$Y[n] = \{1, 5, 5, -5, -6, 4, 1, -2\}$$

5.2 Some Properties of The Z-Transform

Example: (Final 2006) given $h_1[n] = (n - 1)u[n]$,
 $h_2[n] = \delta[n] + nu[n - 1] + \delta[n - 2]$, $h_3 = \left(\frac{1}{2}\right)^n u[n]$

Find the T. F of the System



5.2 Some Properties of The Z-Transform

Solution: $H(z) = \frac{Y(z)}{X(z)} = H_3(z)[H_2(z) - H_1(z)]$

$$H_1(z) = \frac{z}{(z-1)^2} - \frac{z}{z-1} = \frac{2z-z^2}{(z-1)^2}$$

$$H_2(z) = 1 + \left(-z \frac{d}{dz}\right) z^{-1} u(z) + z^{-2} = \frac{z^4 - z^3 + 2z^2 + 1}{z^2(z-1)^2}$$

$$H_3(z) = \frac{z}{z - \frac{1}{2}}$$

$$\therefore H(z) = \frac{2z^3 + z^2 + z - 1}{z(z - \frac{1}{2})(z - 1)}$$

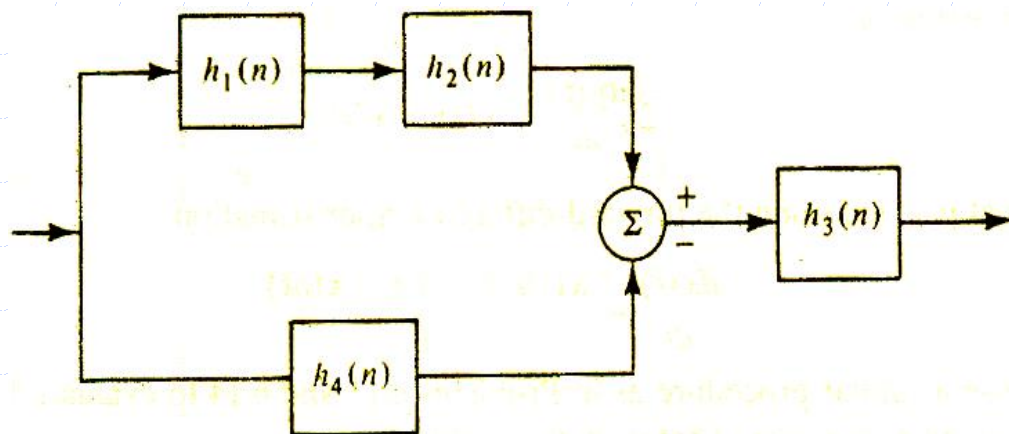
5.2 Some Properties of The Z-Transform

Example (Final 2007) Find the impulse response of the system shown in the Figure below. Assume that

$$h_1[n] = \left(\frac{1}{2}\right)^n u[n]$$

$$h_2[n] = \delta[n]$$

$$h_3[n] = h_4[n] = \left(\frac{1}{3}\right)^n u[n].$$



Solution

$$\begin{aligned} h[n] &= \left[\left(\frac{1}{2}\right)^n u[n] * \delta[n] - \left(\frac{1}{3}\right)^n u[n] \right] * \left(\frac{1}{3}\right)^n u[n] \\ &= \left[\left(\frac{1}{2}\right)^n u[n] * \left(\frac{1}{3}\right)^n u[n] \right] - \left[\left(\frac{1}{3}\right)^n u[n] * \left(\frac{1}{3}\right)^n u[n] \right] \end{aligned}$$

5.2 Some Properties of The Z-Transform

$$\begin{aligned}
 &= \sum_{k=-\infty}^{\infty} \left(\frac{1}{2}\right)^k u(k) \left(\frac{1}{3}\right)^{n-k} u[n-k] - \sum_{k=-\infty}^{\infty} \left(\frac{1}{3}\right)^k u(k) \left(\frac{1}{3}\right)^{n-k} u[n-k] \\
 &= \left(\frac{1}{3}\right)^n \sum_{k=0}^n \left(\frac{3}{2}\right)^k - \left(\frac{1}{3}\right)^n \sum_{k=0}^n (1)^k = \\
 &= \left(\frac{1}{3}\right)^n \left[\frac{\left(\frac{3}{2}\right)^{n+1} - 1}{\frac{3}{2} - 1} \right] - \left(\frac{1}{3}\right)^n \left(1 + \sum_{k=1}^n (1)^k\right) \\
 &= \left[3 \left(\frac{1}{2}\right)^n - 2 \left(\frac{1}{3}\right)^n - \left(\frac{1}{3}\right)^n (n+1) \right] u[n] \\
 &= \left[3 \left(\frac{1}{2}\right)^n - 3 \left(\frac{1}{3}\right)^n - n \left(\frac{1}{3}\right)^n \right] u[n]
 \end{aligned}$$

5.2 Some Properties of The Z-Transform

Example

Suppose that

$$x[n] = 2\delta[n] - 3\delta[n-2] + 4\delta[n-3]$$

$$h[n] = \delta[n] + 2\delta[n-1] + \delta[n-2]$$

We wish to find $y[n]$ by finding first $Y(Z)$

Solution

$$X(z) = 2 - 3z^{-2} + 4z^{-3}$$

$$H(z) = 1 + 2z^{-1} + z^{-2}$$

$$Y(z) = X(z)H(z)$$

$$Y(z) = (2 - 3z^{-2} + 4z^{-3})(1 + 2z^{-1} + z^{-2})$$

$$= 2 + 4z^{-1} - z^{-2} - 2z^{-3} + 5z^{-4} + 4z^{-5}$$

$$y[n] = 2\delta[n] + 4\delta[n-1] - \delta[n-2]$$

$$- 2\delta[n-3] + 5\delta[n-4] + 4\delta[n-5]$$

5.2 Some Properties of The Z Transform

Example Let us find the convolution given by two signals
 $x_1(n) = \{3, -2, 2\}$ $x_2(n) = \{2, 0 \leq n \leq 4 \text{ and } 0 \text{ elsewhere}\}$

Solution

Z-transformation of the first equation can be written as;

$$\sum_{n=-\infty}^{\infty} x_1(n)Z^{-n} = 3 - 2Z^{-1} + 2Z^{-2}$$

Z-transformation of the second signal can be written as;

$$\sum_{n=-\infty}^{\infty} x_2(n)Z^{-n} = 2 + 2Z^{-1} + 2Z^{-2} + 2Z^{-3} + 2Z^{-4}$$

So, the convolution of the above two signals is given by

$$X(Z) = [x_1(Z) * x_2(Z)] = [3 - 2Z^{-1} + 2Z^{-2}] \times [2 + 2Z^{-1} + 2Z^{-2} + 2Z^{-3} + 2Z^{-4}]$$

Taking the inverse Z-transformation $x(n) = \{6, 2, 6, 6, 6, 0, 4\}$

TABLE 10.2 SOME COMMON z -TRANSFORM PAIRS

Signal	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\frac{1}{1 - z^{-1}}$	$ z > 1$
3. $-u[-n - 1]$	$\frac{1}{1 - z^{-1}}$	$ z < 1$
4. $\delta[n - m]$	z^{-m}	All z , except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $\alpha^n u[n]$	$\frac{1}{1 - \alpha z^{-1}}$	$ z > \alpha $
6. $-\alpha^n u[-n - 1]$	$\frac{1}{1 - \alpha z^{-1}}$	$ z < \alpha $
7. $n\alpha^n u[n]$	$\frac{\alpha z^{-1}}{(1 - \alpha z^{-1})^2}$	$ z > \alpha $
8. $-n\alpha^n u[-n - 1]$	$\frac{\alpha z^{-1}}{(1 - \alpha z^{-1})^2}$	$ z < \alpha $
9. $[\cos \omega_0 n]u[n]$	$\frac{1 - [\cos \omega_0]z^{-1}}{1 - [2 \cos \omega_0]z^{-1} + z^{-2}}$	$ z > 1$
10. $[\sin \omega_0 n]u[n]$	$\frac{[\sin \omega_0]z^{-1}}{1 - [2 \cos \omega_0]z^{-1} + z^{-2}}$	$ z > 1$
11. $[r^n \cos \omega_0 n]u[n]$	$\frac{1 - [r \cos \omega_0]z^{-1}}{1 - [2r \cos \omega_0]z^{-1} + r^2 z^{-2}}$	$ z > r$
12. $[r^n \sin \omega_0 n]u[n]$	$\frac{[r \sin \omega_0]z^{-1}}{1 - [2r \cos \omega_0]z^{-1} + r^2 z^{-2}}$	$ z > r$

Properties of the z -Transform

Property	Time Domain	z -Domain
Notation	$x(n)$ $x_1(n)$ $x_2(n)$	$X(z)$ $X_1(z)$ $X_2(z)$
Linearity and Superposition	$a_1 x_1(n) + a_2 x_2(n)$	$a_1 X_1(z) + a_2 X_2(z)$
Time-Shifting	$x(n - k)$	$z^{-k} X(z)$
Scaling in the z -domain	$a^n x(n)$	$X(a^{-1} z)$
Time reversal	$x(-n)$	$X(z^{-1})$
Conjugation	$x^*(n)$	$X^*(z^*)$
Real part	$Re[x(n)]$	$\frac{1}{2} [X(z) + X^*(z^*)]$
Imag part	$Imag[x(n)]$	$\frac{1}{2j} [X(z) - X^*(z^*)]$
Differentiation in the z -domain	$nx(n)$	$-z \frac{d}{dz} X(z)$
Convolution	$x_1(n) \otimes x_2(n)$	$X_1(z) X_2(z)$
Correlation	$r_{x_1 x_2}(l) = x_1(l) \otimes x_2(-l)$	$R_{x_1 x_2}(z) = X_1(z) X_2(z^{-1})$



The Inverse of Z-Transform

There are many methods for finding the inverse of Z-transform; Three methods will be discussed in this class.

1. Direct Division Method (**Power Series Method**)
2. Inversion by Partial fraction Expansion
3. Inversion Integral Method

The Inverse of Z-Transform

1. Direct Division Method (Power Series Method):

The power series can be obtained by arranging the numerator and denominator of $X(z)$ in descending power of Z then divide.

Example determine the inverse Z- transform :

$$X(z) = \frac{Z}{Z - 0.1} \quad |Z| > 0.1$$

Solution

$$\begin{array}{r} 1 + 0.1Z^{-1} + .1^2 Z^{-2} + \dots \\ Z-0.1 \overline{) \begin{array}{l} Z \\ Z-0.1 \\ \hline 0.1 \end{array}} \end{array}$$

$$X(z) = 1 + .1Z^{-1} + .1^2 Z^{-2} + \dots \rightarrow x[n] = (0.1)^n u[n]$$

The Inverse of Z-Transform

Example $X(z) = \frac{z^3 - z^2 - z - \frac{1}{16}}{z^3 - \frac{3}{4}z^2 - \frac{1}{2}z - \frac{1}{16}} \quad |z| > \frac{1}{2} \quad \text{find } x[n]$

Solution

$$\begin{array}{r} 1 + \frac{1}{4}z^{-1} + \frac{13}{16}z^{-2} + \dots \\ z^3 - \frac{3}{4}z^2 - \frac{1}{2}z - \frac{1}{16} \overline{) \quad z^3 - z^2 - z - \frac{1}{16}} \\ \quad \quad \quad z^3 - \frac{5}{4}z^2 - \frac{1}{2}z - \frac{1}{16} \end{array}$$

$X(0)=1, x(1)=1/4, x(2)=13/16, \dots$

In this example, it is not easy to determine the general expression for $x[n]$. As seen, the direct division method may be carried out by hand calculations if only the first several terms of the sequence are desired. **In general the method does not yield a closed form for $x[n]$.**

The Inverse of Z-Transform

2. Inversion by Partial-fraction Expansion

T.F has to be rational function, to obtain the inverse Z transform.

The use of partial fractions here is *almost* exactly the same as for Laplace transforms.....the only difference is that you first divide by z **before** performing the partial fraction expansion...then after expanding you multiply by z to get the final expansion

Example $X(z) = \frac{z^3 - z^2 - z - \frac{1}{16}}{z^3 - \frac{3}{4}z^2 - \frac{1}{2}z - \frac{1}{16}} \quad |Z| > \frac{1}{2} \quad \text{find } x[n]$

The Inverse of Z-Transform

Solution:

$$\frac{X(z)}{Z} = \frac{Z^3 - Z^2 - Z - \frac{1}{16}}{Z(Z^3 - \frac{3}{4}Z^2 - \frac{1}{2}Z - \frac{1}{16})} = \frac{Z^3 - Z^2 - Z - \frac{1}{16}}{Z(Z - \frac{1}{2})^2(Z - \frac{1}{4})}$$

$$\frac{X(z)}{Z} = \frac{A}{Z} + \frac{B}{(Z - \frac{1}{2})^2} + \frac{C}{(Z - \frac{1}{2})} + \frac{D}{Z - \frac{1}{4}}$$

Using same method used in Laplace transform To find A,B,C,D
 $A=1, B=5/2, C=-9, D=9$

$$\therefore \frac{X(z)}{Z} = \frac{1}{Z} + \frac{\frac{5}{2}}{(Z - \frac{1}{2})^2} + \frac{-9}{(Z - \frac{1}{2})} + \frac{9}{Z - \frac{1}{4}}$$

$$X(z) = 1 + \frac{\frac{5}{2}Z}{(Z - \frac{1}{2})^2} + \frac{-9Z}{(Z - \frac{1}{2})} + \frac{9Z}{Z - \frac{1}{4}}$$

$$x[n] = \delta[n] + [5n(\frac{1}{2})^n + 9(\frac{1}{4})^n - 9(\frac{1}{2})^n u[n]]$$

The Inverse of Z-Transform

- Obtain the inverse z-transform of the function

$$F(z) = \frac{z+1}{z^2+0.3z+0.02}$$

Solution: Solve using two different methods.

(i) Partial Fraction Expansion (dividing by z)

$$\frac{F(z)}{z} = \frac{z+1}{z(z+0.1)(z+0.2)} = \frac{A}{z} + \frac{B}{z+0.1} + \frac{C}{z+0.2}$$
$$F(z) = \frac{Az}{z} + \frac{Bz}{z+0.1} + \frac{Cz}{z+0.2}$$

The Inverse of Z-Transform

$$A = z \frac{F(z)}{z} \Bigg|_{z=0} = F(0) = \frac{1}{0.02} = 50$$

$$B = (z + 0.1) \frac{F(z)}{z} \Bigg|_{z=-0.1} = \frac{1 - 0.1}{(-0.1)(0.1)} = -90$$

$$C = (z + 0.2) \frac{F(z)}{z} \Bigg|_{z=-0.2} = \frac{1 - 0.2}{(-0.2)(-0.1)} = 40$$

- Partial fraction expansion then multiply by z

$$F(z) = 50 - \frac{90z}{z + 0.1} + \frac{40z}{z + 0.2}$$

The Inverse of Z-Transform

$$F(z) = 50 - \frac{90z}{z + 0.1} + \frac{40z}{z + 0.2}$$

- (ii) Table Lookup

$$f(k) = \begin{cases} 50\delta(k) - 90(-0.1)^k + 40(-0.2)^k, & k \geq 0 \\ 0, & k < 0 \end{cases}$$

- **Note:** $f(0) = 50 - 90 + 40 = 0$
- The time sequence can be rewritten as

$$f(k) = \begin{cases} -90(-0.1)^k + 40(-0.2)^k, & k \geq 1 \\ 0, & k < 1 \end{cases}$$

The Inverse of Z-Transform

- (i) Partial Fraction Expansion (without dividing by z)

$$F(z) = \frac{z + 1}{(z + 0.1)(z + 0.2)} = \frac{A}{z + 0.1} + \frac{B}{z + 0.2}$$

- Partial fraction coefficients

$$A = (z + 0.1)F(z)\big|_{z=-0.1} = \frac{1 - 0.1}{0.1} = 9$$

$$B = (z + 0.2)F(z)\big|_{z=-0.2} = \frac{1 - 0.2}{-0.1} = -8$$

The Inverse of Z-Transform

- Partial Fraction Expansion

$$\begin{aligned} F(z) &= \frac{9}{z + 0.1} - \frac{8}{z + 0.2} \\ &= \frac{9z}{z + 0.1} z^{-1} - \frac{8z}{z + 0.2} z^{-1} \end{aligned}$$

- (ii) Table Lookup (use the delay theorem)

$$f(k) = \begin{cases} 9(-0.1)^{k-1} - 8(-0.2)^{k-1}, & k \geq 1 \\ 0, & k < 1 \end{cases}$$

(Verify: same answer as before)

The Inverse of Z-Transform

MATLAB

$G(z)$ numerator $z + 1$,
denominator $z^3 + 0.3z^2 + 0.02z$

```
>> num = [1, 1]
```

```
>> den = [1, 0.3, 0.02, 0]
```

- Partial Fraction Coefficients

```
>> [r, p, k] = residue( num, den)
```

- **p** = poles, **r** = residues
- **k** = coefficients of remainder polynomial.

The Inverse of Z-Transform

Results

r =

40.0000

-90.0000

50.0000

p =

-0.2000

-0.1000

0

k =

[]

The Inverse of Z-Transform

Obtain the inverse z-transform of the function

$$F(z) = \frac{1}{z^2(z - 0.5)}$$

Solution

(i) *Partial Fraction Expansion* (Dividing by z)

$$\frac{F(z)}{z} = \frac{1}{z^3(z - 0.5)} = \frac{A_{11}}{z^3} + \frac{A_{12}}{z^2} + \frac{A_{13}}{z} + \frac{A_4}{z - 0.5}$$

$$A_4 = (z - 0.5) \left. \frac{F(z)}{z} \right|_{z=0.5} = \left. \frac{1}{z^3} \right|_{z=0.5} = 8$$

The Inverse of Z-Transform

$$A_{11} = z^3 \frac{F(z)}{z} \Big|_{z=0} = \frac{1}{z - 0.5} \Big|_{z=0} = -2$$

$$A_{12} = \frac{1}{1!} \frac{d}{dz} z^3 \frac{F(z)}{z} \Big|_{z=0} = \frac{d}{dz} \frac{1}{z - 0.5} \Big|_{z=0}$$

$$= \frac{-1}{(z - 0.5)^2} \Big|_{z=0} = -4$$

$$A_{12} = \frac{1}{2!} \frac{d^2}{dz^2} z^3 \frac{F(z)}{z} \Big|_{z=0} = \left(\frac{1}{2} \right) \frac{d}{dz} \frac{-1}{(z - 0.5)^2} \Big|_{z=0}$$

$$= \left(\frac{1}{2} \right) \frac{(-1)(-2)}{(z - 0.5)^3} \Big|_{z=0} = -8$$

The Inverse of Z-Transform

- Partial Fraction Expansion

$$F(z) = \frac{1}{z^2(z - 0.5)} = \frac{8z}{z - 0.5} - 2z^{-2} - 4z^{-1} - 8$$

- (ii) *Table Lookup*
- z-transform tables and Definition 2.1 yield

$$\begin{aligned} f(k) &= \begin{cases} 8(0.5)^k - 2\delta(k - 2) - 4\delta(k - 1) - 8\delta(k), & k \geq 0 \\ 0, & k < 0 \end{cases} \end{aligned}$$

The Inverse of Z-Transform

Example 5.3 P 501 given $X(z) = \frac{2Z(3Z+17)}{(Z-1)(Z^2-6Z+25)}$

Find the inverse Z-Transform.

Solution: $\frac{X(z)}{Z} = \frac{2(3Z+17)}{(Z-1)(Z-3-j3)(Z-3+j4)}$

$$\frac{X(z)}{Z} = \frac{k_1}{Z-1} + \frac{k_2}{Z-3-j4} + \frac{k_2^*}{Z-3+j4}$$

$$X(z) = 2 \frac{Z}{Z-1} + \frac{1.6e^{-j2.246}Z}{Z-3-j4} + \frac{1.6e^{j2.246}Z}{Z-3+j4}$$

From Table 5-1 (12-b)

$$Z^{-1} \left\{ \frac{(0.5re^{j\theta})Z}{Z-\gamma} + \frac{(0.5re^{j\theta})Z}{Z-\gamma^*} \right\} = r|\gamma|^n \cos(\beta_n + \theta) u[n]$$

The Inverse of Z-Transform

$$\gamma = |\gamma|e^{j\beta} \quad 0.5r=1.6 \quad \longrightarrow \quad r=3.2, \theta=-2.246 \text{ rad}$$

$$\gamma=3+j4=5 e^{j0.927} \quad |\gamma| = 5, \beta = 0.927$$

$$x[n] = [2 + 3.2(5)^n \cos(0.927n - 2.246)]u[n]$$

Example find $y[n]$

$$Y(z) = \frac{z+1}{z^2 + \frac{3}{4}z + \frac{1}{8}}$$

First divide $Y(z)$ by z to get:

$$\frac{Y(z)}{z} = \frac{z+1}{z^3 + \frac{3}{4}z^2 + \frac{1}{8}z}$$

Then we have:

$$\frac{Y(z)}{z} = \frac{4}{z + \frac{1}{2}} - \frac{12}{z + \frac{1}{4}} + \frac{8}{z} \quad \longrightarrow \quad Y(z) = \frac{4z}{z + \frac{1}{2}} - \frac{12z}{z + \frac{1}{4}} + 8$$

$$\longrightarrow y[n] = 4\left(-\frac{1}{2}\right)^n u[n] - 12\left(-\frac{1}{4}\right)^n u[n] + 8\delta[n]$$

The Inverse of Z-Transform

3. Inversion integral Method:

$$x[n] = \frac{1}{2\pi j} \oint X(z) Z^{n-1} dz$$

$$x[n] = \sum_{\substack{\text{at poles} \\ \text{of } X(z)Z^{n-1}}} [\text{residue of } X(z)Z^{n-1}]$$

If the function $X(z) Z^{n-1}$ has a simple pole at $Z=a$ then the residue is evaluated as

$$[\text{residue}]|_{z=a} = (Z - a)X(z)Z^{n-1}|_{z=a}$$

The Inverse of Z-Transform

For a pole of order m at $Z=a$ the residue is calculated using the following expression:

$$[residue]|_{Z=a} = \frac{1}{(m-1)!} \frac{d^{m-1}}{dz} [(Z-a)^m X(z) Z^{n-1}]|_{Z=a}$$

Example Find $x[n]$ for $X(z) = \frac{z}{(z-1)^2}$

Solution: The only method to solve above function is by integral method.

$X(z)Z^{n-1}$ has multiple poles at $Z=1$

The Inverse of Z-Transform

$$X(z)Z^{n-1} = \frac{Z}{(Z-1)^2} Z^{n-1} = \frac{Z^n}{(Z-1)^2}$$

$$\therefore x[n] = \frac{1}{(2-1)!} \frac{d^{2-1}}{dz^{2-1}} [(Z-1)^2 \frac{Z}{(Z-1)^2} Z^{n-1}]|_{z=1}$$

$$x[n] = \frac{d}{dz} (Z^n)|_{z=1} = nZ^{n-1}|_{z=1} = n$$

Example Obtain the inverse Z transform of

$$X(z) = \frac{Z^{-2}}{(1-Z^{-1})^3} \quad ROC \quad |Z| > 1$$

Solution: $X(z) = \frac{Z}{(Z-1)^3}$

$$X(z)Z^{n-1} = \frac{Z}{(Z-1)^3} Z^{n-1} = \frac{Z^n}{(Z-1)^3}$$

The Inverse of Z-Transform

$X(z)Z^{n-1}$ has a triple pole at $Z=1$

$x[n] = [\text{residue of } \frac{Z^n}{(z-1)^3} \text{ at triple pole } Z=1]$

$$x[n] = \frac{1}{(3-1)!} \frac{d^2}{dz^2} [(z-1)^3 \frac{Z^n}{(Z-1)^3}]|_{Z=1}$$

$$x[n] = \frac{1}{2} n(n-1) Z^{n-2}|_{Z=1} = \frac{1}{2} n(n-1) \quad n = 0, 1, 2, \dots$$

Example Obtain the inverse Z transform of

$$X(z) = \frac{Z(Z+2)}{(Z-1)^2}$$

The Inverse of Z-Transform

By Partial Fraction:

$$\frac{X(z)}{z} = \frac{Z + 2}{(Z - 1)^2} = \frac{k_1}{(Z - 1)^2} + \frac{k_2}{Z - 1}$$

$$k_1 = 3, \quad k_2 = 1 \quad X(z) = \frac{3Z}{(Z-1)^2} + \frac{Z}{Z-1}$$

$$x[n] = [3n + 1]u[n]$$

By Inversion Integral Method:

$$X(z)Z^{n-1} = \frac{(Z+2)Z^n}{(Z-1)^2}, \quad X(z)Z^{n-1} \text{ has double poles at } Z=1$$

$$x[n] = [\text{residue of } \frac{(Z+2)Z^n}{(Z-1)^2} \text{ at double poles at } Z=1]$$

$$\therefore x[n] = \frac{1}{(2-1)!} \frac{d}{dz} \left[(Z-1)^2 \frac{(Z+2)Z^n}{(Z-1)^2} \right] \Big|_{Z=1} = \frac{d}{dz} [(Z+2)Z^n] \Big|_{Z=1}$$

$$x[n] = 3n + 1 \quad n = 0, 1, 2, \dots$$

Example

Obtain the inverse z transform of

$$X(z) = \frac{1}{(z-1)^2 z^2}$$

Solution:

$$x(z) z^{k-1} = \frac{z^{k-1}}{(z-1)^2 z^2} = \frac{1}{(z-1)^2 z^{3-k}}$$

$x(z) z^{k-1}$ has double pole at $z=1$ and $(3-k)$ multiple pole at $z=0$

Example

for $k=0$ double pole at $z=1$ and triple pole at $z=0$
 $k=1$ " " " double " "
 $k=2$ " " " simple " "
 $k=3, 4, 5, \dots$ " " " no " "

Example

For $k=0$

$$x(0) = \left[\text{residue of } \frac{1}{(z-1)^2 z^3} \text{ at } z=1 \right] +$$
$$\left[\text{residue of } \frac{1}{(z-1)^2 z^3} \text{ at triple } z=0 \right]$$

$$= \frac{1}{(2-1)!} \lim_{z \rightarrow 1} \frac{d}{dz} \left[(z-1)^2 \frac{1}{(z-1)^2 z^3} \right] + \frac{1}{(3-1)!} \lim_{z \rightarrow 0}$$

$$\frac{d^2}{dz^2} \left[z^3 \frac{1}{(z-1)^2 z^3} \right] = -3 + 3 = 0$$

Example

For $k=1$

$$x(1) = \frac{1}{(2-1)!} \lim_{z \rightarrow 0} \frac{d}{dz} \left[(z-1)^2 \frac{1}{(z-1)^2} z^2 \right] +$$

$$\frac{1}{(2-1)!} \lim_{z \rightarrow 0} \frac{d}{dz} \left[z^2 \frac{1}{(z-1)^2} z^2 \right] = -2 + 2 = 0$$

For $k=2$

$$x(2) = -1 + 1 = 0$$

For $k=3, 4, 5, \dots$

$$x(k) = \frac{1}{(2-1)!} \lim_{z \rightarrow 0} \frac{d}{dz} \left[(z-1)^2 \frac{z^{k-3}}{(z-1)^2} \right] = k-3$$

Example

$$X[k] = \begin{cases} 0 & k = 0, 1, 2, 3 \\ k - 3 & k = 4, 5, 6, \dots \end{cases}$$

The Inverse of Z-Transform

Example

Find $Z^{-1}[X(z)]$, where

$$X(z) = \frac{z^2}{(z-1)^2(z-e^{-aT})}$$

Solution:

$$X(z)z^{k-1} = \frac{z^{k+1}}{(z-1)^2(z-e^{-aT})}$$

Simple pole at $z = e^{-aT}$

Double pole at $z = 1$

$$\begin{aligned} x(k) &= \sum_{i=1}^2 \left[\text{residue of } \frac{z^{k+1}}{(z-1)^2(z-e^{-aT})} \text{ at pole } z = z_i \right] \\ &= K_1 + K_2 \end{aligned}$$

The Inverse of Z-Transform

where

$$K_1 = \lim_{z \rightarrow e^{-aT}} \left[(z - e^{-aT}) \frac{z^{k+1}}{(z-1)^2(z - e^{-aT})} \right] = \frac{e^{-a(k+1)T}}{(1 - e^{-aT})^2}$$

$$K_2 = \frac{1}{(2-1)!} \lim_{z \rightarrow 1} \frac{d}{dz} \left[(z-1)^2 \frac{z^{k+1}}{(z-1)^2(z - e^{-aT})} \right]$$

$$= \frac{k}{1 - e^{-aT}} - \frac{e^{-aT}}{(1 - e^{-aT})^2} \rightarrow \text{see steps below}$$

$$\Rightarrow x(kT) = \frac{kT}{T(1 - e^{-aT})} - e^{-aT} \frac{(1 - e^{-akT})}{(1 - e^{-aT})^2} \quad k = 0, 1, 2, \dots$$

The Inverse of Z-Transform

Steps

$$\begin{aligned} d\frac{v}{u} &= \frac{u dv - v du}{u^2} \\ \lim_{z \rightarrow 1} \frac{d}{dz} \left(\frac{z^{k+1}}{z - e^{-aT}} \right) &= \lim_{z \rightarrow 1} \frac{(k+1)z^k(z - e^{-aT}) - z^{k+1}}{(z - e^{-aT})^2} \\ &= \lim_{z \rightarrow 1} \left[\frac{(k+1)z^k}{z - e^{-aT}} - \frac{z^{k+1}}{(z - e^{-aT})^2} \right] \\ &= \frac{k+1}{1 - e^{-aT}} - \frac{1}{(1 - e^{-aT})^2} \\ &= \frac{k}{1 - e^{-aT}} + \frac{1 - e^{-aT}}{(1 - e^{-aT})^2} - \frac{1}{(1 - e^{-aT})^2} \\ &= \frac{k}{1 - e^{-aT}} - \frac{e^{-aT}}{(1 - e^{-aT})^2} \end{aligned}$$

Property Name	Property	
Linearity	$ax[n] + bv[n]$	$aX(z) + bV(z)$
Right Time Shift (Causal Signal)	$x[n - q], \quad q > 0$	$z^{-q}X(z)$
Right Time Shift (Non-Causal Signal)	$x[n - 1]$ $x[n - 2]$ $x[n - q], \quad q > 0$	$z^{-1}X(z) + x[-1]$ $z^{-2}X(z) + x[-2] + z^{-1}x[-1]$ $z^{-q}X(z) + x[-q] + z^{-1}x[-q + 1] + \dots$ $\dots + z^{-q+1}x[-1]$
Multiply by n	$nx[n]$	$-z \frac{d}{dz} X(z)$
Multiply by n^2	$n^2 x[n]$	$z \frac{d}{dz} X(z) + z^2 \frac{d^2}{dz^2} X(z)$
Multiply by Exponential	$a^n x[n], \quad a \text{ real or complex}$	$X(z/a), \quad a \text{ real or complex}$
Multiply by Sine	$\sin(\Omega_0 n)x[n]$	$\frac{j}{2} [X(e^{j\Omega_0} z) - X(e^{-j\Omega_0} z)]$
Multiply by Cosine	$\cos(\Omega_0 n)x[n]$	$\frac{1}{2} [X(e^{j\Omega_0} z) + X(e^{-j\Omega_0} z)]$
Summation (Causal Signal)	$\sum_{i=0}^n x[i]$	$\frac{z}{z-1} X(z)$
Convolution in Time	$x[n] * h[n]$	$X(z)H(z)$
Initial-Value Theorem	$x[0] = \lim_{z \rightarrow \infty} [X(z)]$ $x[1] = \lim_{z \rightarrow \infty} [zX(z) - zx[0]]$ $x[q] = \lim_{z \rightarrow \infty} [z^q X(z) - z^q x[0] - z^{q-1} x[1] - \dots - zx[q-1]]$	
Final-Value Theorem	If $X(z)$ is rational and the poles of $(z-1)X(z)$ are inside unit circle Then $\lim_{n \rightarrow \infty} x[n] = [(z-1)X(z)]_{z=1}$	

Transfer Function

$$y[n] = y[n-1] + u[n]$$

output $\nearrow$ $\nwarrow$ input

Z-transform $\rightarrow Y(z) = y[-1] + z^{-1}Y(z) + U(z)$

$\nwarrow$ initial condition

$$(1 - z^{-1})Y(z) = y[-1] + U(z)$$

$$Y(z) = \frac{y[-1]}{1 - z^{-1}} + \frac{U(z)}{1 - z^{-1}}$$

Zero-Input
Response $\nearrow$

$\nwarrow$ Zero-state
Response with
transfer function

$$H(z) = \frac{1}{1 - z^{-1}}$$

Transfer Function

$$y[n] = 0.5 y[n-1] + 0.5 y[n-2] + u[n]$$

$$y[n] \xrightarrow{Z} Y(z)$$

$$u[n] \xrightarrow{Z} U(z)$$

$$0.5 y[n-1] \xrightarrow{Z} 0.5 y[-1] + 0.5 z^{-1} Y(z)$$

$$0.5 y[n-2] \xrightarrow{Z} \text{Apply the delay rule twice !}$$

$$0.5 y[-2] + 0.5 z^{-1} Z\{y[n-1]\}$$

$$= 0.5 y[-2] + 0.5 z^{-1} y[-1] + 0.5 z^{-1} z^{-1} Y(z)$$

$$= 0.5 y[-2] + 0.5 z^{-1} y[-1] + 0.5 z^{-2} Y(z)$$

Transfer Function

$$Y(z) = 0.5y[-1] + 0.5z^{-1}Y(z) + 0.5y[-2] + 0.5z^{-1}y[-1] + 0.5z^{-2}Y(z) + U(z)$$

$$Y(z) = \frac{0.5y[-1] + 0.5y[-2] + 0.5z^{-1}y[-1]}{1 - 0.5z^{-1} - 0.5z^{-2}} +$$

$$\frac{1}{1 - 0.5z^{-1} - 0.5z^{-2}} U(z)$$

Zero Input Response

Zero State Response

ZT For Difference Eqs.

Given a difference equation that models a D-T system we may want to solve it:

-with IC's

Apply ZT to the Difference Equation

-with IC's of zero

Use the Transfer Function Approach

Note...the ideas here are very much like what we did with the Laplace Transform for CT systems.

We'll consider the ZT/Difference Eq. approach first...

Solving a First-order Difference Equation using the ZT

Given: $y[n] + ay[n-1] = bx[n]$

IC = $y[-1]$

$x[n]$ for $n = 0, 1, 2, \dots$

Solve for: $y[n]$ for $n = 0, 1, 2, \dots$

Take ZT of differential equation: $Z\{y[n] + ay[n-1]\} = Z\{bx[n]\}$ Use Linearity of ZT

$$Z\{y[n]\} + aZ\{y[n-1]\} = bZ\{x[n]\}$$

$Y(z)$

$X(z)$

Need Right-Shift Property...
but which one???

Because of the non-zero IC we need to use the non-causal form:

$$Z\{y[n-1]\} = z^{-1}Y(z) + y[-1]$$

Solving a First-order Difference Equation using the ZT

Using these results gives:

$$Y(z) + a[z^{-1}Y(z) + y[-1]] = bX(z)$$

Which is an algebraic equation that can be solved for $Y(z)$:

$$Y(z) = \frac{-ay[-1]}{1 + az^{-1}} + \frac{b}{1 + az^{-1}} X(z)$$

Not the best form for doing Inverse ZT... we want things in terms of z not z^{-1}

Multiply each term by z/z

$$Y(z) = -ay[-1] \frac{z}{z+a} + \frac{bz}{z+a} X(z)$$

On ZT Table

Part due to input signal modified by **Transfer Function**

$$H(z) = \frac{bz}{z+a}$$

$$y[n] = -ay[-1](-a)^n u[n] + Z^{-1}\{H(z)X(z)\}$$

If $|a| < 1$ this dies out as $n \uparrow$, its an IC-driven transient

If the ICs are zero, this is all we have!!!

First Order System w/ Step Input

For $x[n] = u[n] \leftrightarrow X(z) = \frac{z}{z-1}$

Then using our general results we just derived we get:

$$Y(z) = \frac{-ay[-1]z}{z+a} + \left(\frac{bz}{z+a} \right) \left(\frac{z}{z-1} \right)$$

For now assume that $a \neq -1$ so we don't have a repeated root.

Then doing Partial Fraction Expansion we get (and we have to do the PFE by hand because we don't know a ... but it is not that hard!!!)

$$Y(z) = \frac{-ay[-1]z}{z+a} + \frac{\left(\frac{ab}{a+1}\right)z}{z+a} + \frac{\left(\frac{b}{a+1}\right)z}{z-1}$$

Now using
ZT Table
we get:

$$y[n] = -ay[-1](-a)^n + \frac{b}{a+1} [a(-a)^n + (1)^n] \quad n = 0, 1, 2, \dots$$

IC-Driven Transient:
decays if system is stable

Input-Driven Output... 2 Terms:

1st term decays (Transient)

2nd term persists (Steady State)

Solving a Second-order Difference Equation using the ZT

The Given Difference Equation: $y[n] + a_1y[n-1] + a_2y[n-2] = b_0x[n] + b_1x[n-1]$

Assume that the input is causal

Assume you are given ICs: $y[-1]$ & $y[-2]$

Find the system response $y[n]$ for $n = 0, 1, 2, 3, \dots$

Take the ZT using the non-causal right-shift property:

$$Y(z) + a_1(z^{-1}Y(z) + y[-1]) + a_2(z^{-2}Y(z) + z^{-1}y[-1] + y[-2]) = b_0X(z) + b_1z^{-1}X(z)$$

$$Y(z) = \frac{-a_1y[-1] - a_2y[-1]z^{-1} - a_2y[-2]}{1 + a_1z^{-1} + a_2z^{-2}} + \underbrace{\frac{b_0 + b_1z^{-1}}{1 + a_1z^{-1} + a_2z^{-2}}}_{H(z)} X(z)$$

**Due to IC's... decays
if system is stable**

$H(z)$

**Due to input – will have
transient part and
steady-state part**

Solving a Nth-order Difference Equation using the ZT

$$y[n] + \sum_{i=1}^N a_i y[n-i] = \sum_{i=0}^M b_i x[n-i]$$

Contains $x[n], x[n-1], \dots$

If this system is causal, we won't have $x[n+1], x[n+2], \dots$ here

$$A(z) = z^N + a_1 z^{N-1} + \dots + a_{N-1} z + a_N$$

$$B(z) = b_0 z^N + b_1 z^{N-1} + \dots + b_M z^{N-M}$$

$C(z)$ = depends on the IC's

Transforming gives:

$$Y(z) = \frac{C(z)}{A(z)} + \underbrace{\frac{B(z)}{A(z)}}_{H(z)} X(z)$$

Transient and steady state part due to input

$H(z)$ – transfer function

Transient part due to ICs

Example

System described by a difference equation

$$y[n] - 5y[n-1] + 6y[n-2] = 3f[n-1] + 5f[n-2]$$

$$y[-1] = 11/6, \quad y[-2] = 37/36$$

$$f[n] = 2^{-n} u[n] \quad \text{find } y[n]$$

Example

$$Y[z] - 5\left(\frac{1}{z}Y[z] + y[-1]\right) + 6\left(\frac{1}{z^2}Y[z] + \frac{1}{z}y[-1] + y[-2]\right) = 3\left(\frac{1}{z}F[z] + f[-1]\right) + 5\left(\frac{1}{z^2}F[z] + \frac{1}{z}f[-1] + f[-2]\right)$$

$$\left(1 - \frac{5}{z} + \frac{6}{z^2}\right)Y[z] - \left(3 - \frac{11}{z}\right) = \frac{3}{z-0.5} + \frac{5}{z(z-0.5)}$$

$$Y[z] = \frac{26}{15}\left(\frac{z}{z-0.5}\right) - \frac{7}{3}\left(\frac{z}{z-2}\right) + \frac{18}{5}\left(\frac{z}{z-3}\right)$$

$$y[n] = \left[\frac{26}{15}(0.5)^n - \frac{7}{3}(2)^n + \frac{18}{5}(3)^n \right] u[n]$$

Example

Find the unit impulse response of the system described by the following equation :

$$y[n+2] + 2y[n+1] + y[n] = 2x[n+2] - x[n+1]$$

Solution:

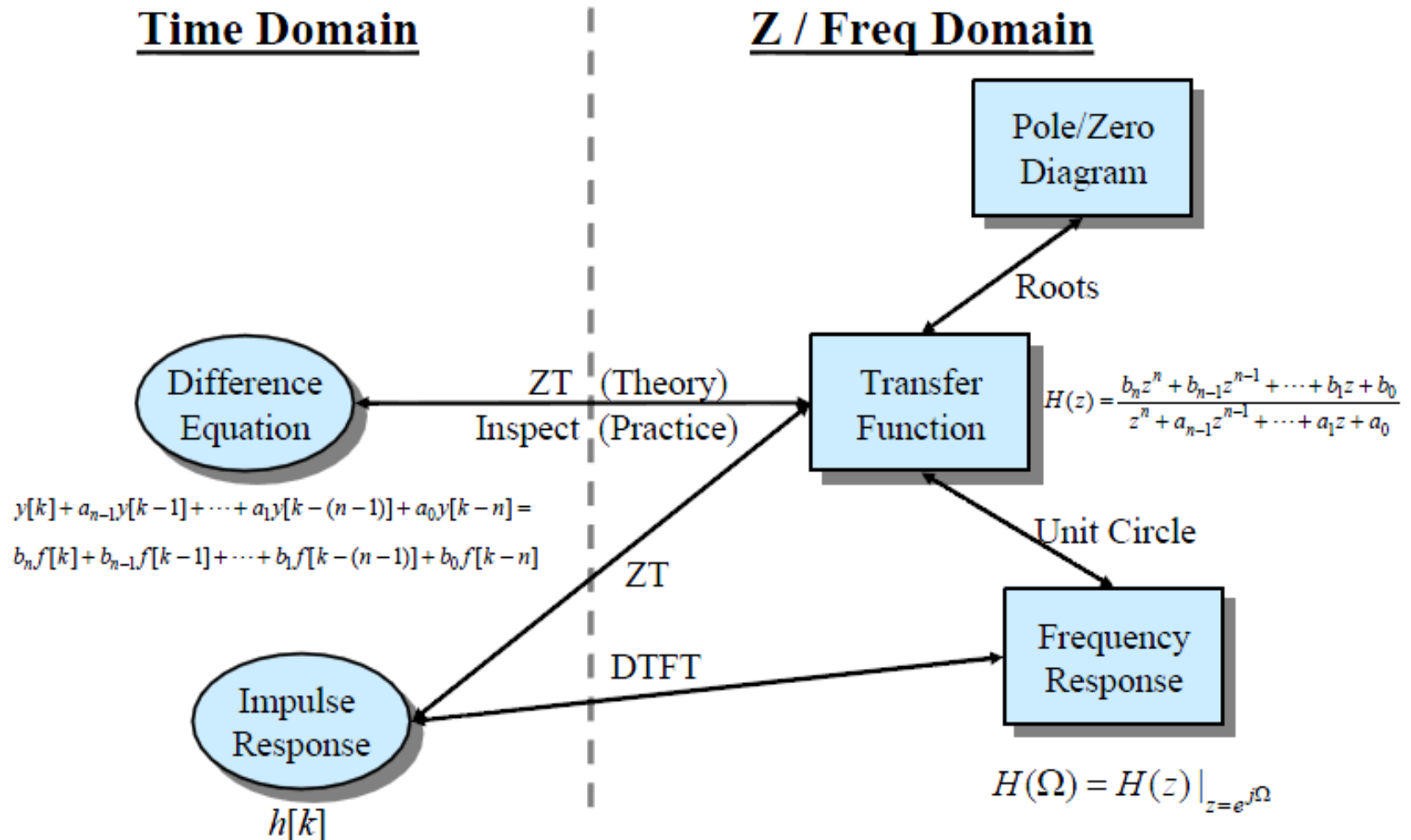
$$H(z) = \frac{2z^2 - z}{z^2 + 2z + 1} = \frac{z(2z - 1)}{(z + 1)^2}$$

Example

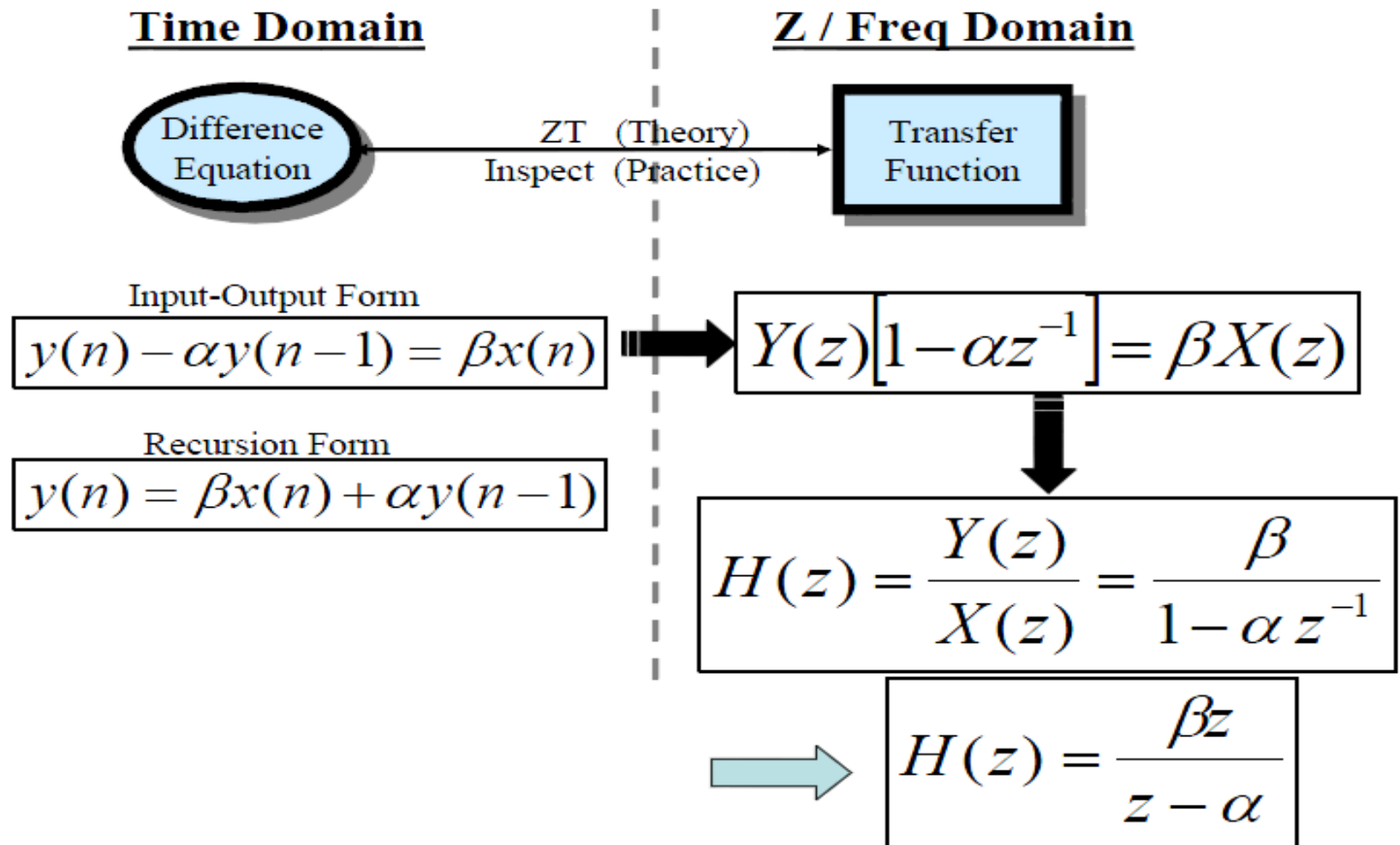
$$H(z) = \frac{2}{z+1} - \frac{3}{(z+1)^2} \Rightarrow H(z) = 2 \left(\frac{z}{z+1} \right) - 3 \frac{z}{(z+1)^2}$$

$$h[n] = [2(-1)^n + 3n(-1)^n] u[n] = (2+3n)(-1)^n u[n]$$

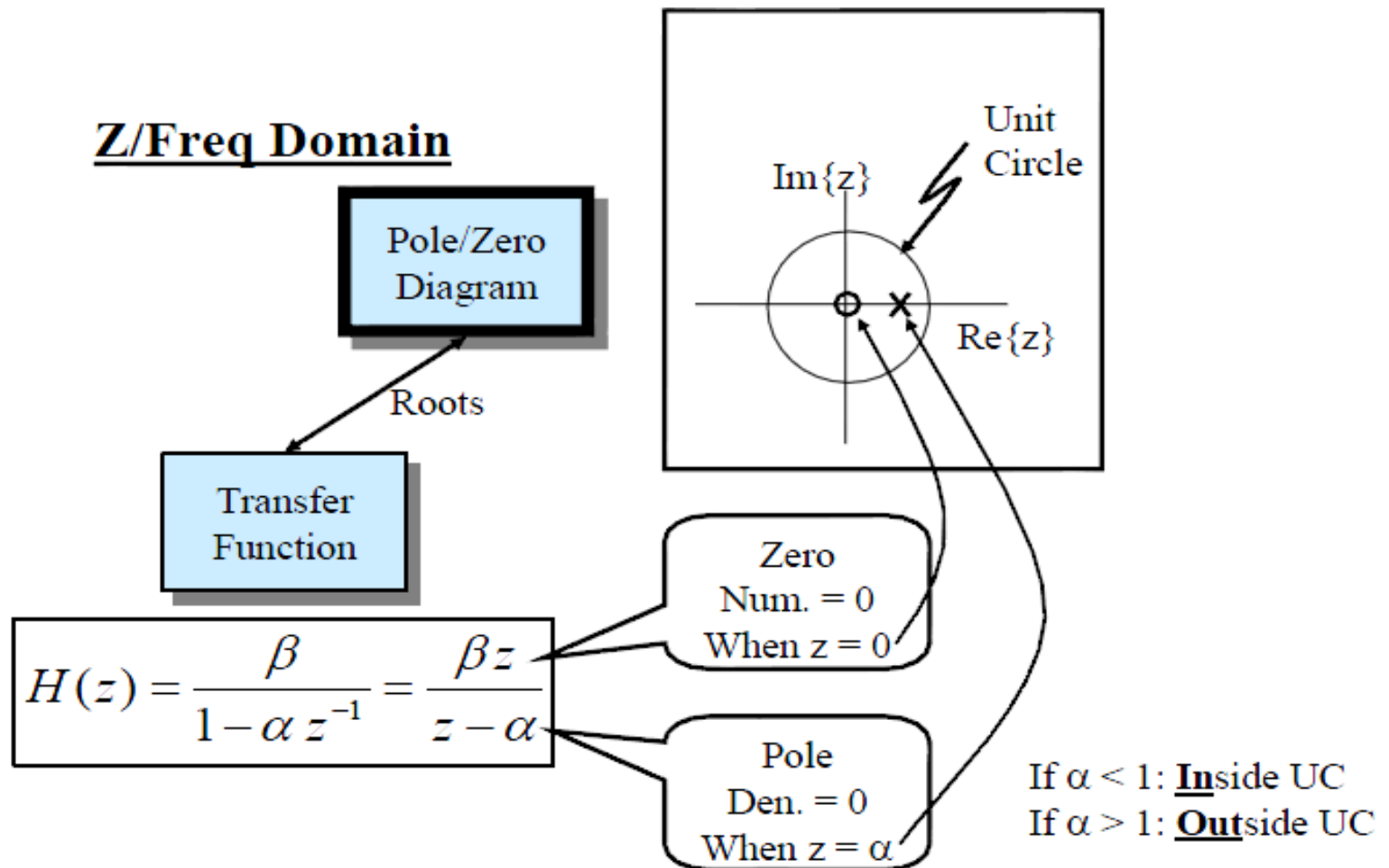
Discrete-Time System Relationships



Example System Relationships



Example System Relationships



Example

A system with impulse response $h(n) = 2\left(\frac{1}{3}\right)^n u(n-1)$ produces an output $y(n) = (-2)^n u(n-1)$.

Determine the corresponding input $x(n)$.

Solution

$$h[n] = 2 \frac{1}{3} \left(\frac{1}{3}\right)^{n-1} u[n-1]$$
$$y[n] = -2(-2)^{n-1} u[n-1]$$

Solution

$$H(z) = z^{-1} \frac{2z/3}{z - 1/3}$$

$$Y(z) = z^{-1} \frac{-2z}{z+2}$$

$$Y(z) = H(z) X(z) \Rightarrow X(z) = \frac{Y(z)}{H(z)} = \frac{z^{-1} \frac{-2z}{z+2}}{z^{-1} \frac{2z/3}{z - 1/3}}$$

$$\therefore X(z) = -3 \frac{z - 1/3}{z+2} = -3 \left[\frac{z}{z+2} - \frac{1/3}{z+2} \right]$$

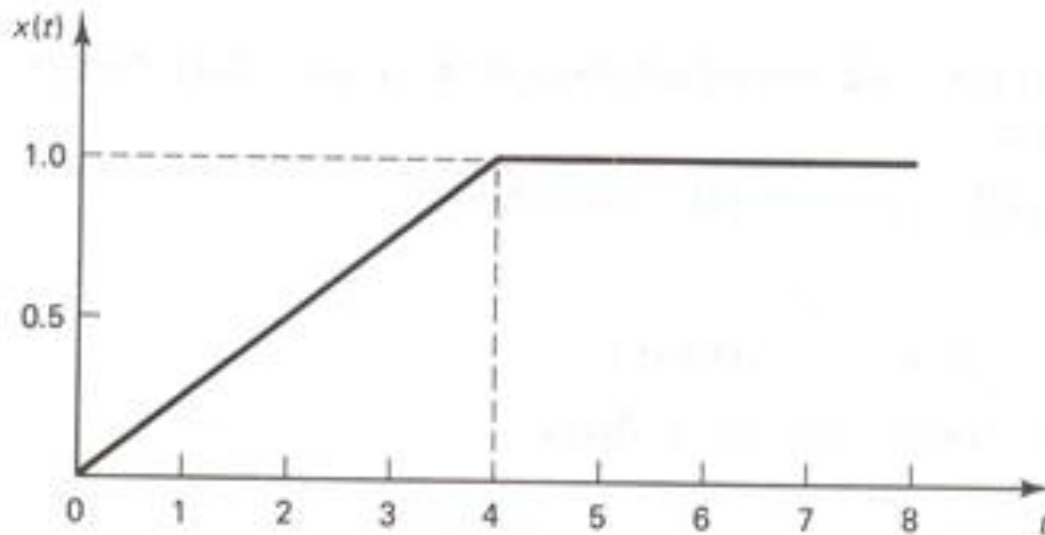
$$\therefore x[n] = -3 \left[(-2)^n u[n] - 1/3 (-2)^{n-1} u[n-1] \right]$$

OR

$$x[n] = \frac{1}{2} \delta[n] - \frac{7}{2} (-2)^n u[n]$$

Example

Obtain the z transform of the curve shown in Figure Assume $T = 1$ sec



Solution

$$\begin{aligned}x(0) &= 0 \\x(1) &= 0.25 \\x(2) &= 0.50 \\x(3) &= 0.75 \\x(k) &= 1\end{aligned}$$

$$k = 4, 5, 6, \dots$$

$$X(z) = \sum_{k=0}^{\infty} x(k) z^{-k}$$

$$= 0.25 z^{-1} + 0.5 z^{-2} + 0.75 z^{-3} + z^{-4} + z^{-5} + \dots$$

$$= 0.25 (z^{-1} + 2z^{-2} + 3z^{-3}) + z^{-4} \frac{1}{1-z^{-1}}$$

$$X(z) = \frac{z^{-1} + z^{-2} + z^{-3} + z^{-4}}{4(1-z^{-1})} \cdot \frac{(1-z^{-1})}{(1-z^{-1})}$$

$$X(z) = \frac{1}{4} \frac{z^{-1}(1-z^{-4})}{(1-z^{-1})^2}$$

Example

Solve $y[n+2] - 2y[n+1] + 2y[n] = x[n]$

with $y[-1]=1$, $y[-2]=0$, and $x[n]=u[n]$.

delay form is

$$y[n] - 2y[n-1] + 2y[n-2] = x[n-2]$$

$$Y(z) - 2[z^{-1}Y(z) + 1] + 2[z^{-2}Y(z) + z^{-1}] = z^{-2} \frac{z}{z-1}$$

$$\left[\frac{z^2 - 2z + 2}{z^2} \right] Y(z) = \frac{2z^2 - 4z + 3}{z(z-1)}$$

Example

$$\therefore \frac{Y'(z)}{z} = \frac{2z^2 - 4z + 3}{(z-1)(z^2 - 2z + 2)} = \frac{1}{(z-1)} + \frac{z-1}{z^2 - 2z + 2}$$

$$\therefore Y'(z) = \frac{z}{z-1} + \frac{z(z-1)}{z^2 - 2z + 2}$$

$$\therefore y[n] = [1 + (\sqrt{2})^n \cos(\frac{\pi}{4}n)] u[n]$$