

Signal & Linear system

Chapter 6 CT Signal Analysis : Fourier Series

Basil Hamed

Why do We Need Fourier Analysis?

- The essence of Fourier analysis is to represent periodic signals in terms of complex exponentials (or trigonometric functions)
- Many reasons:
 - Almost any signal can be represented as a series of complex exponentials
 - Response of an LTI system to a complex exponential is also a complex exponential with a scaled magnitude.
 - A compact way of approximating several signals. This opens a lot of applications:
 - storing analog signals (such as music) in digital environment
 - over a digital network, transmitting digital equivalent of the signal instead of the original analog signal is easier!

Jean Baptiste Joseph Fourier

Jean Baptiste Joseph Fourier (21 March 1768 – 16 May 1830) was a French mathematician and physicist born in Auxerre and best known for initiating the investigation of **Fourier series** and their applications to problems of heat transfer and vibrations. **The Fourier transform** and Fourier's Law are also named in his honor.



Fourier went with Napoleon Bonaparte on his Egyptian expedition in 1798, and was made governor of Lower Egypt and secretary of the Institut d'Égypte

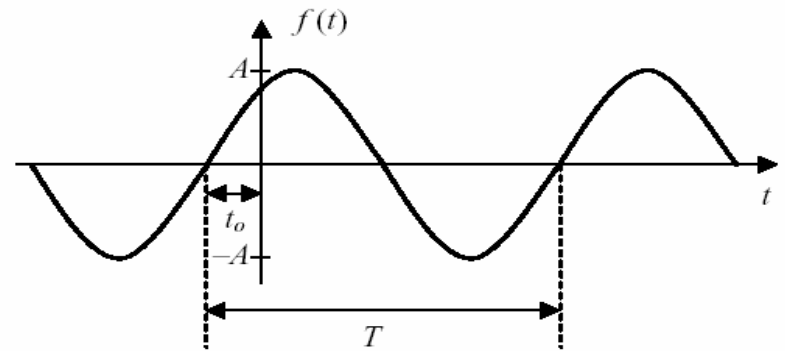
6.1 Periodic Signal Representation By Trigonometric Fourier Series

Fourier Series relates to periodic functions and states that any periodic function can be expressed as the sum of sinusoids(or exponential)

Example
of periodic signal:

Sinusoids

$$f(t) = A \sin(\omega t + \phi)$$



A sinusoid is completely defined by its three parameters:

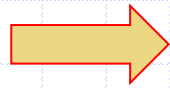
- Amplitude A (for EE's typically in volts or amps or other physical unit)
- Frequency ω in radians per second
- Phase shift ϕ in radians

T is the period of the sinusoid and is related to the frequency

6.1 Periodic Signal Representation By Trigonometric Fourier Series

“Time-domain” model

Given time-domain
signal model $x(t)$



“Frequency-domain model”

Find the Fs coefficients
 $\{a_0, a_n, b_n\}$

Converting “time-domain” signal model into
a “frequency-domain” signal model

6.1 Periodic Signal Representation By Trigonometric Fourier Series

- **General representation of a periodic signal**

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)$$

- **Fourier series coefficients**

$$a_0 = \frac{1}{T_0} \int_0^{T_0} f(t) dt$$

$$a_n = \frac{2}{T_0} \int_0^{T_0} f(t) \cos(n\omega_0 t) dt$$

$$b_n = \frac{2}{T_0} \int_0^{T_0} f(t) \sin(n\omega_0 t) dt$$

Existence of the Fourier Series

- **Existence**

$$\int_0^{T_0} |f(t)| dt < \infty$$

- **Finite number of maxima and minima in one period of $f(t)$**

Dirichlet conditions

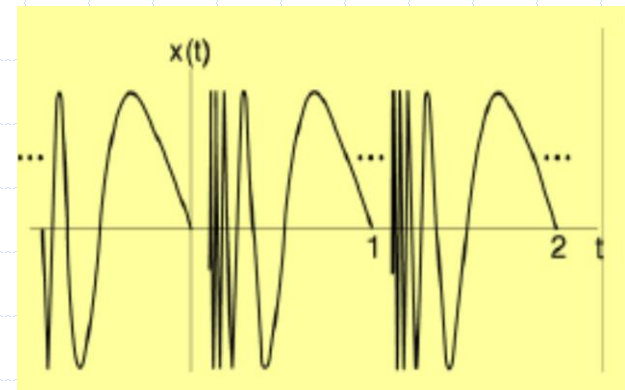
Condition 1. $x(t)$ is absolutely integrable over one period, i. e.

$$\int_T |x(t)| dt < \infty$$

Condition 2. In a finite time interval, $x(t)$ has a *finite* number of maxima and minima

Ex. An example that violates Condition 2.

$$x(t) = \sin\left(\frac{2\pi}{t}\right) \quad 0 < t \leq 1$$



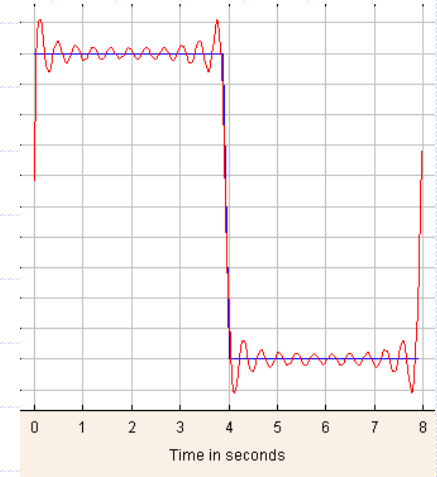
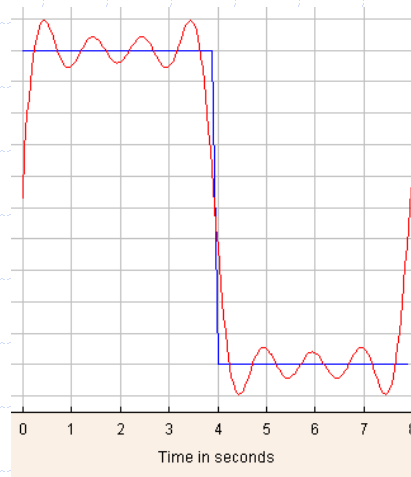
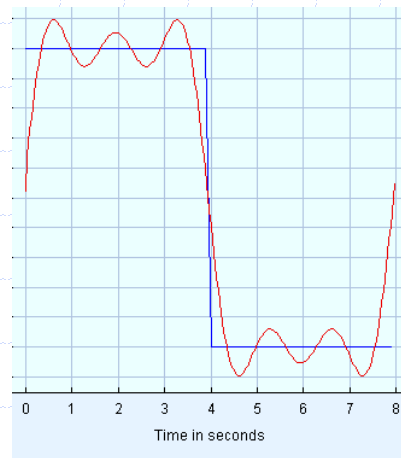
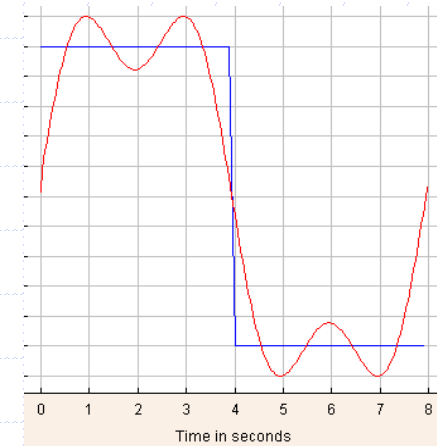
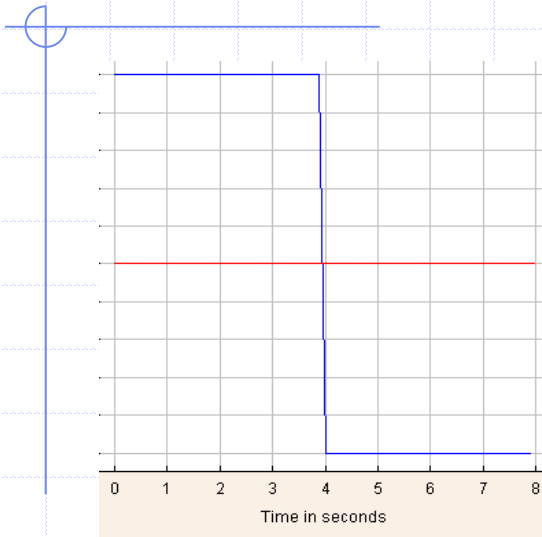
Condition 3. In a finite time interval, $x(t)$ has only a finite number of discontinuities.

Ex. An example that violates Condition 3.

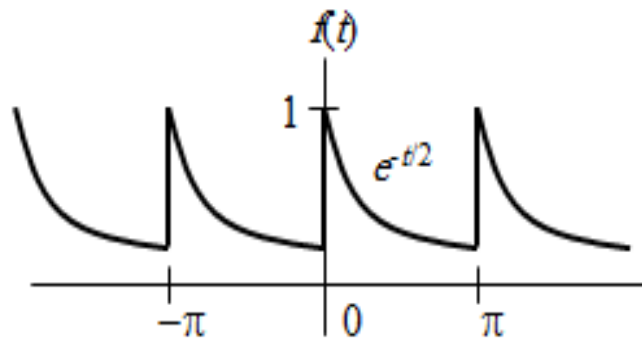


$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)$$

How Fourier Series Works



Example 6.1 P 600



Fundamental period

$$T_0 = \pi$$

Fundamental frequency

$$f_0 = 1/T_0 = 1/\pi \text{ Hz}$$

$$\omega_0 = 2\pi/T_0 = 2 \text{ rad/s}$$

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(2nt) + b_n \sin(2nt)$$

$$a_0 = \frac{1}{\pi} \int_0^{\pi} e^{-\frac{t}{2}} dt = -\frac{2}{\pi} \left(e^{-\frac{\pi}{2}} - 1 \right) \approx 0.504$$

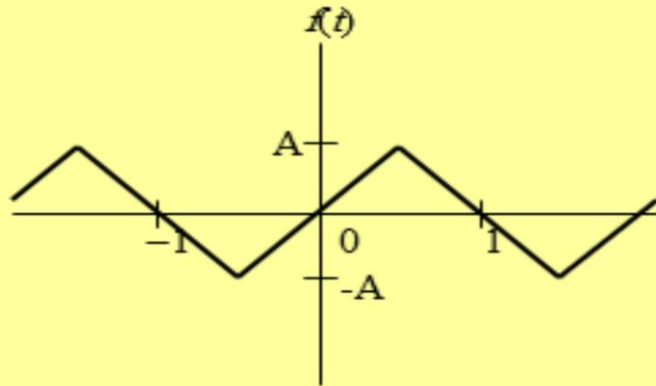
$$a_n = \frac{2}{\pi} \int_0^{\pi} e^{-\frac{t}{2}} \cos(2nt) dt = 0.504 \left(\frac{2}{1+16n^2} \right)$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} e^{-\frac{t}{2}} \sin(2nt) dt = 0.504 \left(\frac{8n}{1+16n^2} \right)$$

a_n and b_n decrease in amplitude as $n \rightarrow \infty$.

$$f(t) = 0.504 \left[1 + \sum_{n=1}^{\infty} \frac{2}{1+16n^2} (\cos(2nt) + 4n \sin(2nt)) \right]$$

Example 6.2 P 604



- Fundamental period**

$$T_0 = 2$$

- Fundamental frequency**

$$f_0 = 1/T_0 = 1/2 \text{ Hz}$$

$$\omega_0 = 2\pi/T_0 = \pi \text{ rad/s}$$

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(\pi n t) + b_n \sin(\pi n t)$$

$$a_0 = 0 \quad (\text{by inspection of the plot})$$

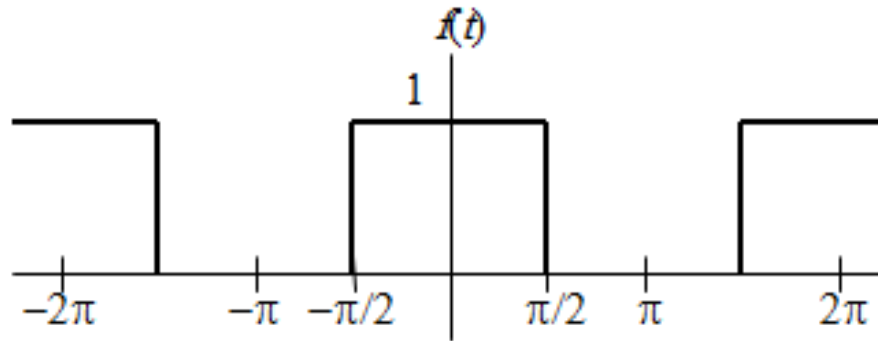
$$a_n = 0 \quad (\text{because it is odd symmetric})$$

$$b_n = \frac{2}{\pi} \int_{-1/2}^{1/2} 2A t \sin(\pi n t) dt +$$

$$\frac{2}{\pi} \int_{1/2}^{3/2} (2A - 2A t) \sin(\pi n t) dt$$

$$b_n = \begin{cases} 0 & n \text{ is even} \\ \frac{8A}{n^2 \pi^2} & n = 1, 5, 9, 13, \dots \\ -\frac{8A}{n^2 \pi^2} & n = 3, 7, 11, 15, \dots \end{cases}$$

Example 6.3 P 6.6



- **Fundamental period**

$$T_0 = 2\pi$$

- **Fundamental frequency**

$$f_0 = 1/T_0 = 1/2\pi \text{ Hz}$$

$$\omega_0 = 2\pi/T_0 = 1 \text{ rad/s}$$

$F(t)$ Over one period:

$$f(t) = \begin{cases} 1 & -\frac{\pi}{2} < t \leq \frac{\pi}{2} \\ 0 & \frac{\pi}{2} < t \leq \frac{3\pi}{2} \end{cases}$$

Example 6.3 P 6.6

Need to find a_0 , b_n , a_n :

$$a_0 = \frac{1}{T_0} \int_0^{T_0} f(t) dt$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} dt = \frac{1}{2}$$

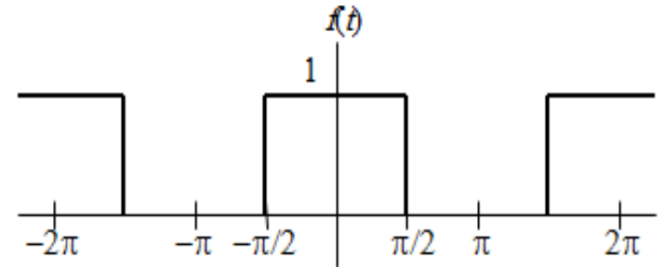
$$a_n = \frac{2}{T_0} \int_0^{T_0} f(t) \cos(n\omega_0 t) dt$$

$$a_n = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \cos nt dt = \frac{2}{n\pi} \sin\left(\frac{n\pi}{2}\right)$$

$$= \begin{cases} 0 & n \text{ even} \\ \frac{2}{\pi n} & n = 1, 5, 9, 13, \dots \\ -\frac{2}{\pi n} & n = 3, 7, 11, 15, \dots \end{cases}$$

$$b_n = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \sin nt dt = 0$$

$$f(t) = \frac{1}{2} + \frac{2}{\pi} \left(\cos t - \frac{1}{3} \cos 3t + \frac{1}{5} \cos 5t - \frac{1}{7} \cos 7t + \dots \right)$$



The Exponential Fourier Series

The periodic function can be expressed using sine and cosine functions, such expressions, however, are not as convenient as the expression using complex exponentials.

$$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

$$\text{Where } C_n = \frac{1}{T} \int_{t_0}^{t_0+T} f(t) e^{-jn\omega_0 t} dt$$

$$a_0 = c_0$$

$$a_n = 2\operatorname{Re}\{C_n\},$$

$$b_n = -2\operatorname{Im}\{C_n\}$$

$$C_n = \frac{1}{2}(a_n - jb_n)$$

The Exponential Fourier Series

Example

Find Fourier Series
Using exponential

Solution

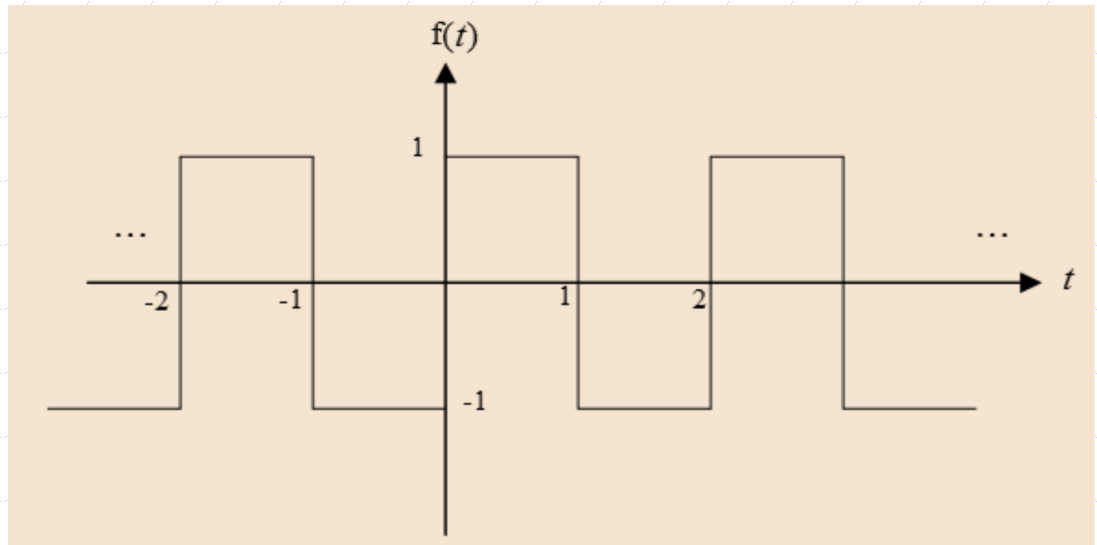
$$T = 2, \omega_0 = \pi$$

Over one period:

$$f(t) = \begin{cases} -1 & -1 \leq t < 0 \\ 1 & 0 \leq t < 1 \end{cases}$$

$$C_n = \frac{1}{T} \int_{t_0}^{t_0+T} f(t) e^{-jn\omega_0 t} dt$$

$$= \frac{1}{2} \left[\int_{-1}^0 -1 e^{-jn\pi t} dt + \int_0^1 1 e^{-jn\pi t} dt \right]$$



The Exponential Fourier Series

$$C_n = \frac{1}{j2n\pi} [2 - e^{jn\pi} - e^{-jn\pi}]$$

$$C_n = \frac{1}{jn\pi} [1 - \cos n\pi] = \frac{j}{\pi} \left[\frac{\cos n\pi - 1}{n} \right]$$

$$\begin{aligned} f(t) &= \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t} \\ &= \sum_{n=-\infty}^{\infty} \frac{j}{n\pi} [\cos n\pi - 1] e^{jn\pi t} \end{aligned}$$

The Exponential Fourier Series

Example

Find Fourier Series
Using exponential

Solution

$$T = 4, \omega_0 = \frac{\pi}{2}$$

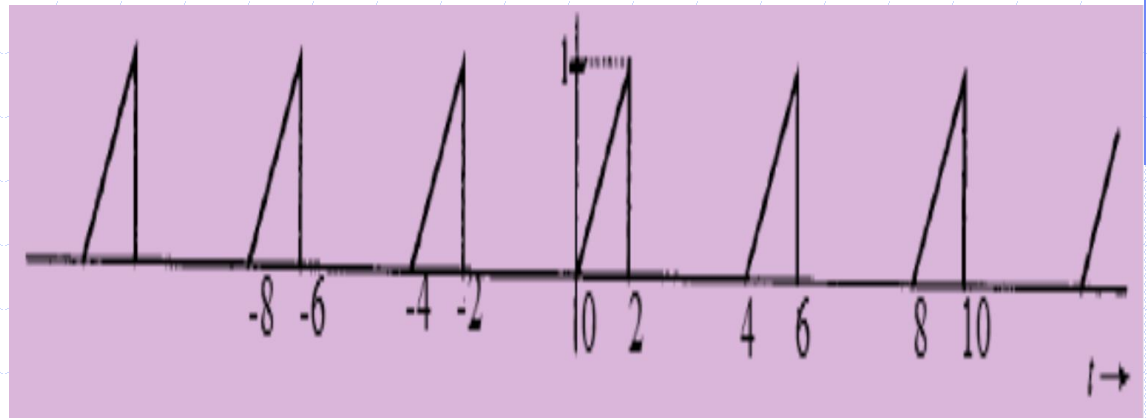
Over one period:

$$f(t) = \begin{cases} \frac{1}{2}t & 0 \leq t < 2 \\ 0 & 2 \leq t < 4 \end{cases}$$

$$C_n = \frac{1}{T} \int_{t_0}^{t_0+T} f(t) e^{-jn\omega_0 t} dt = \frac{1}{4} \int_0^2 \frac{1}{2} t e^{-jn\frac{\pi}{2}t} dt$$

$$C_n = \frac{1}{2n^2\pi^2} [e^{jn\pi} (1 + jn\pi) - 1]$$

Basil Hamed



The Exponential Fourier Series

$$e^{jn\pi} = \cos n\pi + j \sin n\pi$$

$$C_n = \frac{1}{2n^2\pi^2} [\cos n\pi - 1] + j \frac{1}{2} \frac{\cos n\pi}{n\pi} \quad n \neq 0$$

$$C_0 = \frac{1}{4} \int_0^2 \frac{1}{2} t e^{-j0\frac{\pi}{2}t} dt = \frac{1}{4} \int_0^2 \frac{1}{2} t dt = \frac{1}{4}$$

$$f(t) = \frac{1}{4} + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} C_n e^{jn\frac{\pi}{2}t}$$

Line Spectra: (Amplitude Spectrum & Phase Spectrum)

The complex exponential Fourier series of a signal consists of a summation of phasor.

The periodic signal to be characterized graphically in the frequency domain is, by making 2 plots.

The first, showing amplitude versus frequency is known as amplitude spectrum of the signal.

$$C_n = |C_n|e^{j\theta_n} = |C_n|C_n^0 \quad \text{Polar Form}$$

The amplitude spectrum is the plot of $|C_n|$ versus $\omega = n\omega_0$

The second, showing the phase of each component versus frequency is called the phase spectrum of the signal.

The phase spectrum is the plot of the C_n^0 versus $\omega = n\omega_0$

Line Spectra: (Amplitude Spectrum & Phase Spectrum)

$$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

Amplitude spectra: $|C_n| = |C_{-n}|$ is symmetrical (even function)

Phase spectra: $\angle C_n = -\angle C_{-n}$ (odd function)

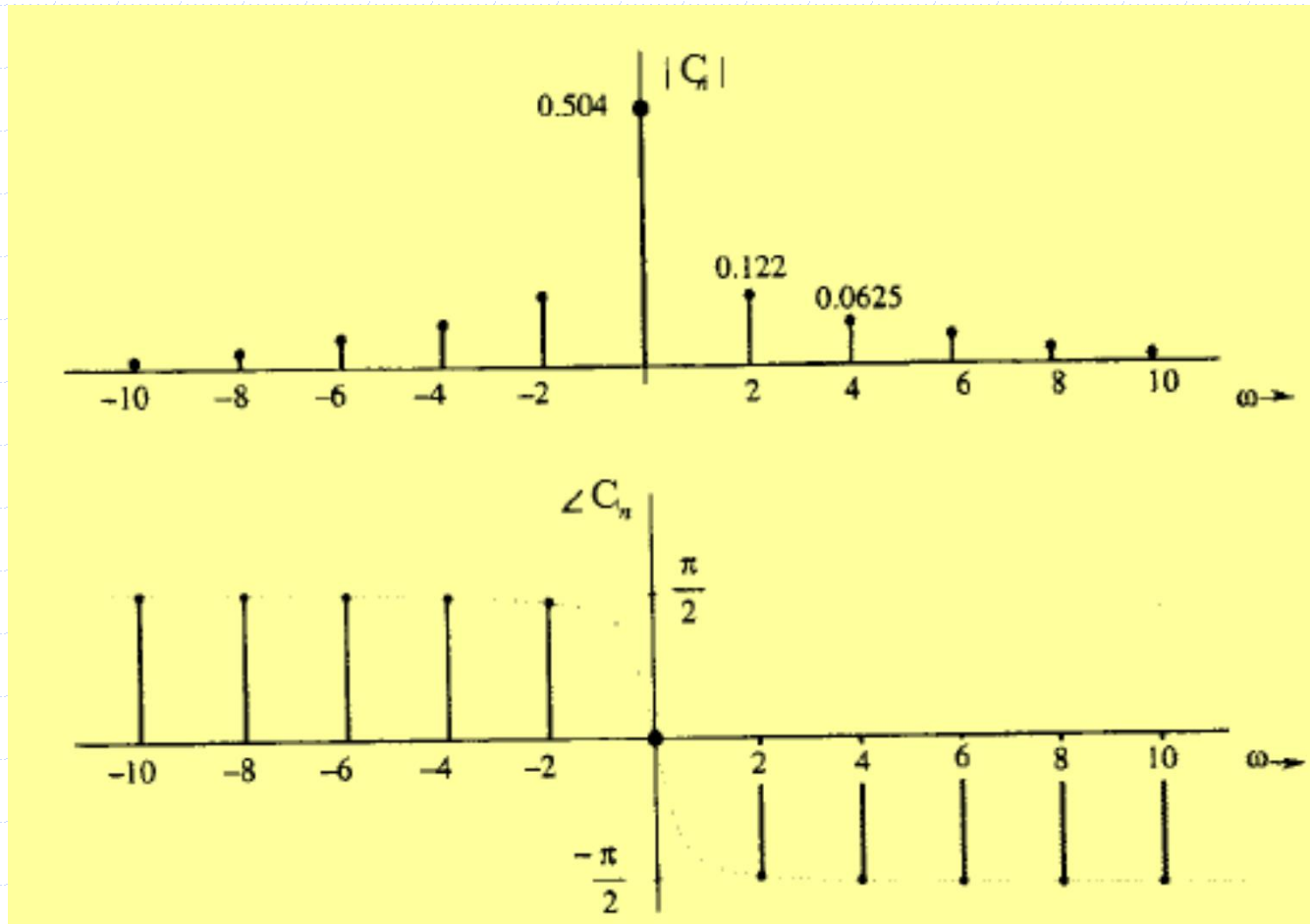
Example $f(t) = \sum_{n=-\infty}^{\infty} \frac{0.504}{1+j4n} e^{jn\omega_0 t}$ Find Line Spectra

Solution: $C_n = \frac{0.504}{1+j4n}$; $C_0 = \frac{0.504}{1+0} = 0.504 \angle 0^\circ$

$$C_1 = \frac{0.504}{1+j4} = 0.122 \angle -75.96^\circ, \quad C_2 = \frac{0.504}{1+j8} = 0.0625 \angle -82.87^\circ$$

$$C_{-1} = \frac{0.504}{1-j4} = 0.122 \angle 75.96^\circ, \quad C_{-2} = \frac{0.504}{1-j8} = 0.0625 \angle 82.87^\circ$$

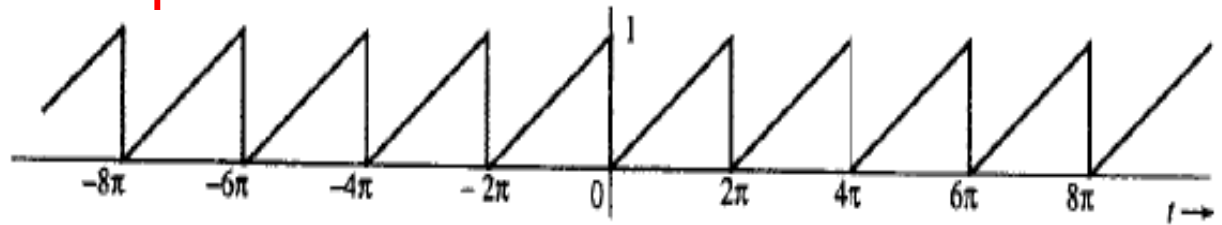
Line Spectra: (Amplitude Spectrum & Phase Spectrum)



Line Spectra: (Amplitude Spectrum & Phase Spectrum)

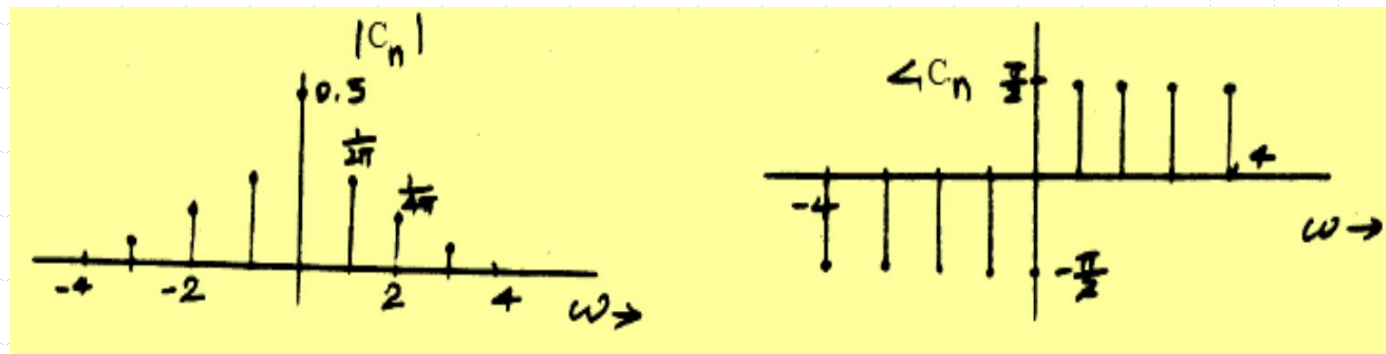
Example: Find the exponential Fourier series and sketch the line spectra

Solution



$$f(t) = C_0 + \sum_{n=-\infty}^{\infty} C_n e^{jnt}, \quad \text{where, by inspection} \quad C_0 = 0.5$$

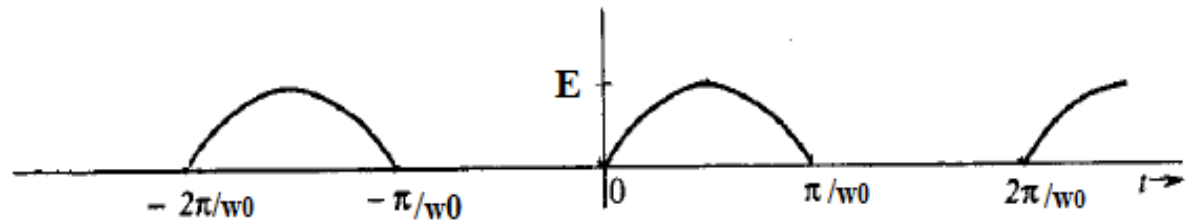
$$C_n = \frac{1}{2\pi} \int_0^{2\pi} \frac{t}{2\pi} e^{-jnt} dt = \frac{j}{2\pi n}, \quad \text{so that} \quad |C_n| = \frac{1}{2\pi n}, \quad \text{and} \quad \angle C_n = \begin{cases} \frac{\pi}{2} & n > 0 \\ -\frac{\pi}{2} & n < 0 \end{cases}$$



Line Spectra: (Amplitude Spectrum & Phase Spectrum)

Example: Find the exponential Fourier series and sketch the line spectra

Solution:



$$x(t) = \begin{cases} E \sin \omega_0 t & 0 < t < \frac{\pi}{\omega_0} \\ 0 & \frac{\pi}{\omega_0} < t < \frac{2\pi}{\omega_0} \end{cases}, \quad x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

$$C_n = \frac{1}{T} \int_0^T x(t) e^{-jn2\frac{\pi}{T}t} dt = \frac{1}{2\pi/\omega_0} \int_0^{\frac{\pi}{\omega_0}} E \sin \omega_0 t e^{-jn2\frac{\pi}{T}t} dt$$

$$C_n = \frac{E e^{-jn\frac{\pi}{2}}}{2\pi(1-n^2)} [e^{-jn\frac{\pi}{2}} + e^{jn\frac{\pi}{2}}] = 2 \cos(n\frac{\pi}{2})$$

Line Spectra: (Amplitude Spectrum & Phase Spectrum)

$$C_n = \frac{E e^{-jn\frac{\pi}{2}}}{\pi(1-n^2)} \cos\left(n\frac{\pi}{2}\right) \quad n \neq \pm 1$$

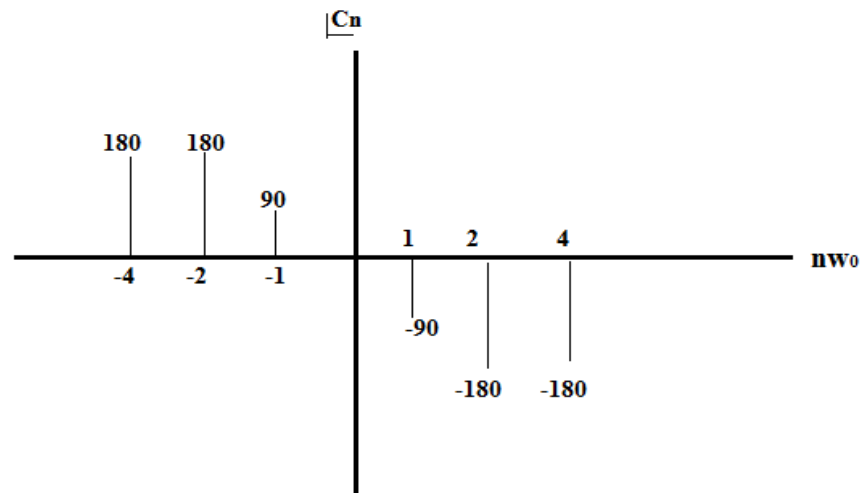
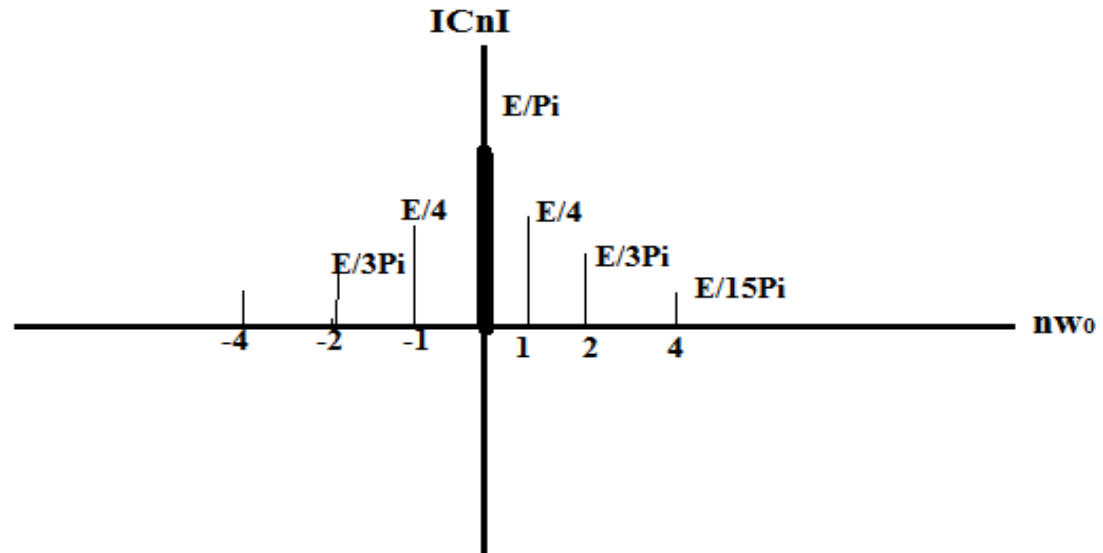
$$C_1 = \frac{1}{2\pi/\omega_0} \int_0^{\frac{\pi}{\omega_0}} E \sin \omega_0 t e^{-jn2\frac{\pi}{T}t} dt = \frac{E}{4j}$$

$$C_n = \begin{cases} \frac{E}{\pi(1-n^2)} & n \text{ even} \\ 0 & n \text{ odd}, n \neq \pm 1 \end{cases}$$

$$C_0 = \frac{E}{\pi}, C_1 = \frac{E}{4j} = \frac{E}{4} \angle -90^\circ, C_2 = \frac{E}{\pi(1-4)} = \frac{E}{3\pi} \angle -180^\circ$$

$$C_4 = \frac{E}{\pi(1-16)} = \frac{E}{15\pi} \angle -180^\circ$$

Line Spectra: (Amplitude Spectrum & Phase Spectrum)



Properties of Fourier series

Effect of waveform symmetry:

1. Even function symmetry $x(t)=x(-t)$

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_n t \quad [b_n = 0]$$

2. Odd function symmetry $x(t)=-x(-t)$

$$f(t) = \sum_{n=1}^{\infty} b_n \sin n\omega_n t \quad [a_n = a_0 = 0]$$

3. Half-Wave odd symmetry $x(t)=-x(t+T/2)=-x(t-T/2)$

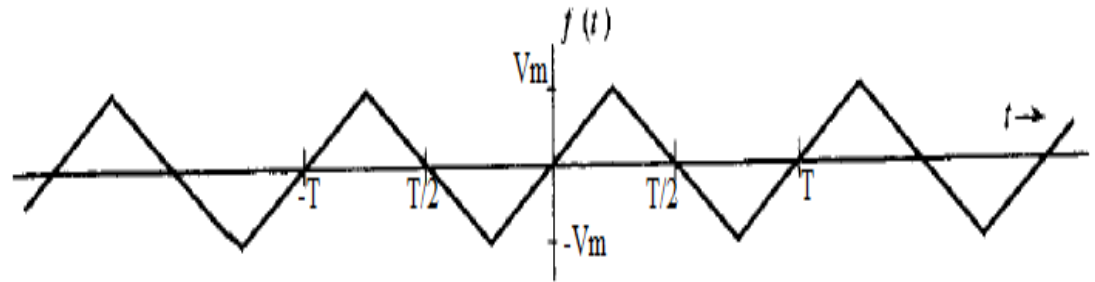
$$a_0=0, a_{2n} = 0, a_{2n+1} \neq 0, b_{2n} = 0, b_{2n+1} \neq 0$$

Remarks: Integrate over $T/2$ only and multiply the coefficient by 2.

Properties of Fourier series

Ex Find Fourier Series

Solution



Function is Odd, Period = T , $\omega_0 = \frac{2\pi}{T}$

$$f(t) = \begin{cases} \frac{4V_m}{T}t & 0 \leq t < \frac{T}{4} \\ -\frac{4V_m}{T}t + 2V_m & \frac{T}{4} \leq t < \frac{T}{2} \end{cases}$$

$$a_n = a_0 = 0$$

Need to find b_n

Properties of Fourier series

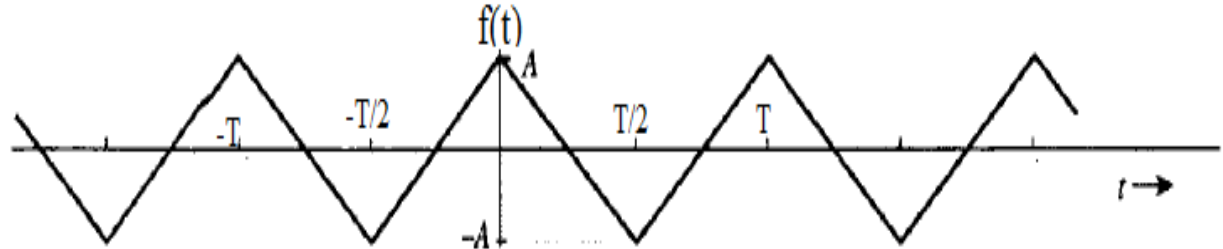
$$b_n = \left[\frac{2}{T} \int_0^{\frac{T}{4}} \frac{4V_m}{T} t \sin n\omega_0 t \, dt + \int_{\frac{T}{4}}^{\frac{T}{2}} \left(\frac{-4}{T} V_m t + 2V_m \right) \sin n\omega_0 t \, dt \right]$$

$$b_n = \frac{8V_m}{\pi^2 n^2} \sin n \frac{\pi}{2} \quad (n \text{ is Odd})$$

$$f(t) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \left(\frac{8V_m}{\pi^2 n^2} \sin n \frac{\pi}{2} \right) \sin \omega_0 t$$

Properties of Fourier series

Ex. Find Fourier series



Solution Function is even Period = T , $\omega_0 = \frac{2\pi}{T}$ $b_n = 0$, $a_0 = 0$

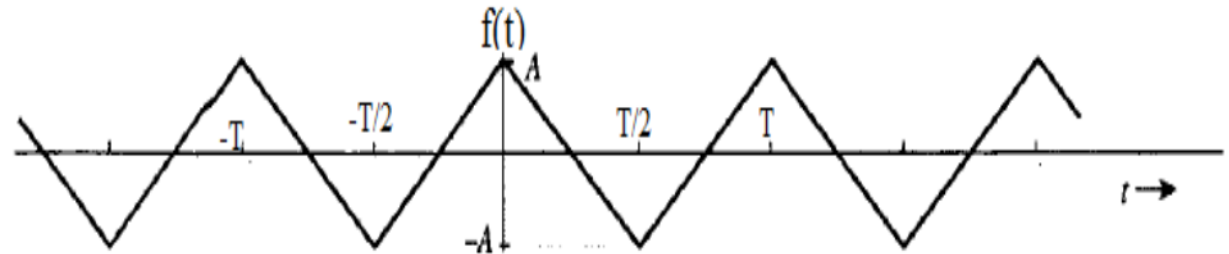
$$f(t) = \begin{cases} A - \frac{4A}{T}t & 0 \leq t < \frac{T}{2} \\ \frac{4A}{T}t - 3A & \frac{T}{2} \leq t < T \end{cases}$$

Need to find a_n

$$a_n = \frac{4}{T} \int_0^{\frac{T}{2}} \left(A - \frac{4A}{T}t \right) \cos n\omega_0 t \, dt = \frac{4A}{(n\pi)^2} (1 - \cos(n\pi))$$

$$a_n = \begin{cases} 0 & n \text{ even} \\ \frac{8A}{(n\pi)^2} & n \text{ odd} \end{cases}$$

Properties of Fourier series



This example is also half-wave

odd symmetry. $x(t) = -x(t + T/2) = -x(t - T/2)$

$$a_0 = 0, a_{2n} = 0, a_{2n+1} \neq 0, b_{2n} = 0, b_{2n+1} \neq 0$$

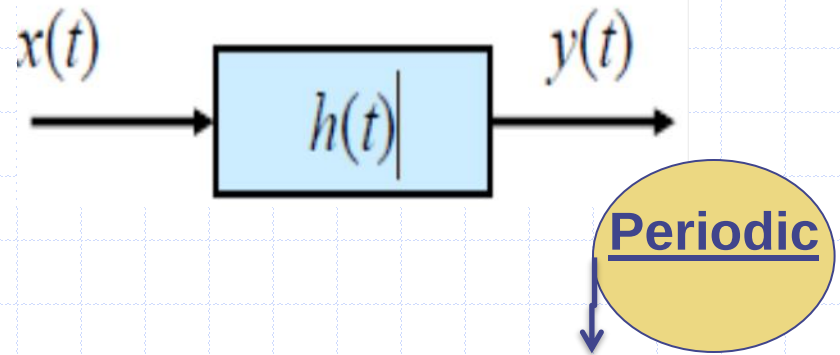
Solution is the same as pervious example

Symmetry	a_0	a_n	b_n	Remarks
Even	$a_0 \neq 0$	$a_n \neq 0$	$b_n = 0$	Integrate over $T/2$ only and
Odd	$a_0 = 0$	$a_n = 0$	$b_n \neq 0$	Multiply by 2
Half-Wave	$a_0 = 0$	$a_{2n} = 0$ $a_{2n+1} \neq 0$	$b_{2n} = 0,$ $b_{2n+1} \neq 0$	

6.4 LTI Systems Response To Periodic Input

Call from Ch# 2:

$$y(t) = \int_{-\infty}^{\infty} x(\tau) h(t - \tau) d\tau$$



For Complex exponential inputs of the form $x(t) = \exp(j\omega t)$

The output of the system is:

$$y(t) = \int_{-\infty}^{\infty} h(\tau) e^{j\omega(t-\tau)} d\tau = e^{j\omega t} \int_{-\infty}^{\infty} h(\tau) e^{-j\omega\tau} d\tau$$

$$\text{Let } H(\omega) = \int_{-\infty}^{\infty} h(\tau) e^{-j\omega\tau} d\tau$$

$$\text{So } y(t) = H(\omega) e^{j\omega t}$$

$H(\omega)$ is called the system T.F and is constant for fixed ω .

6.4 LTI Systems Response To Periodic Input

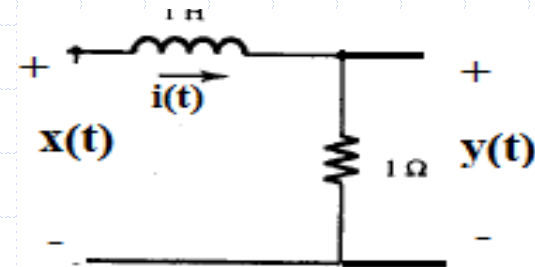
To determine the response $y(t)$ of LTI system to periodic input $x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$, $x(t)$ with the Fourier series representation:

$$y(t) = \sum_{n=-\infty}^{\infty} H(n\omega_0) C_n e^{jn\omega_0 t}$$

Example :

Given $x(t) = 4 \cos t - 2 \cos 2t$

Find $y(t)$



6.4 LTI Systems Response To Periodic Input

Solution KVL $x(t) = L \frac{di}{dt} + y(t)$, $y(t) = Ri(t)$, $\frac{dy(t)}{dt} = R \frac{di(t)}{dt}$

$$x(t) = \frac{L}{R} \frac{dy(t)}{dt} + y(t), \quad \frac{dy}{dt} + \frac{R}{L} y(t) = \frac{R}{L} x(t) \quad (1)$$

$X(t)$ is periodic input:

Set $x(t) = e^{j\omega t}$ (2)

The output voltage is $y(t) = H(\omega) \exp(j\omega t)$ (3)

Sub eq 2&3 into eq 1

$$\begin{aligned} \frac{d}{dt} [H(\omega) e^{j\omega t}] + \frac{R}{L} [H(\omega) e^{j\omega t}] &= \frac{R}{L} e^{j\omega t} = j\omega H(\omega) e^{j\omega t} \\ + \frac{R}{L} H(\omega) e^{j\omega t} &= \frac{R}{L} e^{j\omega t} \end{aligned}$$

$$H(\omega) \left[j\omega + \frac{R}{L} \right] = \frac{R}{L} \quad \text{So} \quad H(\omega) = \frac{R/L}{R/L + j\omega}$$

6.4 LTI Systems Response To Periodic Input

At any frequency $\omega = n\omega_0$ the system T.F:

$$H(n\omega_0) = \frac{R/L}{R/L + jn\omega_0}, \quad \frac{R}{L} = 1, \quad \omega_0 = 1$$

$$y(t) = H(n\omega_0)e^{jn\omega_0 t}, \quad x(t) = 4 \cos t - 2 \cos 2t = 4 \angle 0^\circ - 2 \angle 0^\circ$$

$$y(t) = \frac{1}{1+j1} 4 \angle 0^\circ - \frac{1}{1+j2} 2 \angle 0^\circ$$

$$y(t) = 2\sqrt{2} \angle -45^\circ - \frac{2}{\sqrt{5}} \angle -63^\circ$$

$$y(t) = 2\sqrt{2} \cos(t - 45^\circ) - \frac{2}{\sqrt{5}} \cos(2t - 63^\circ)$$

Why Use Exponentials

The exponential Fourier series is just another way of representing trigonometric Fourier series (or vice versa)

The two forms carry identical information, no more, no less.

Preferring the exponential forms:

- The form is more compact
- LTIC response to exponential signal is also simpler than the system response to sinusoids.
- Much easier to manipulate mathematically.