

Signal & Linear system

Chapter 7 CT Signal Analysis :
Fourier Transform

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Time Domain vs. Frequency Domain

- Fourier Analysis (Series or Transform) is, in fact, a way of determining a given signal's frequency content, i.e. move from time-domain to frequency domain.
- It is always possible to move back from the frequency-domain to time-domain, by either summing the terms of the Fourier Series or by Inverse Fourier Transform.
- Given a signal $x(t)$ in time-domain, its Fourier coefficients (a_k) or its Fourier Transform ($X(\omega)$) are called as its “frequency (or line) spectrum”.

7.1 Aperiodic Signal Representation by Fourier Integral

Recall: Fourier Series represents a periodic signal as a sum of sinusoids

or complex sinusoids $e^{jk\omega_0 t}$

Q: Can we modify the FS idea to handle non-periodic signals?

A: Yes!!

How about:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

Yes...this will work for any *practical* non-periodic signal!!

Called the “Fourier Integral” also, more commonly, called the “**Inverse Fourier Transform**”

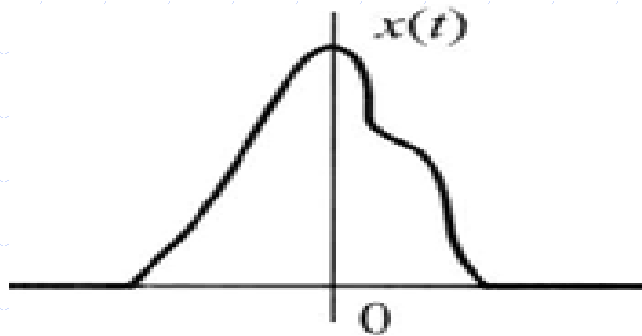
Plays the role of c_k

Plays the role of $e^{jk\omega_0 t}$

Integral replaces sum because it can “add up over the continuum of frequencies”!

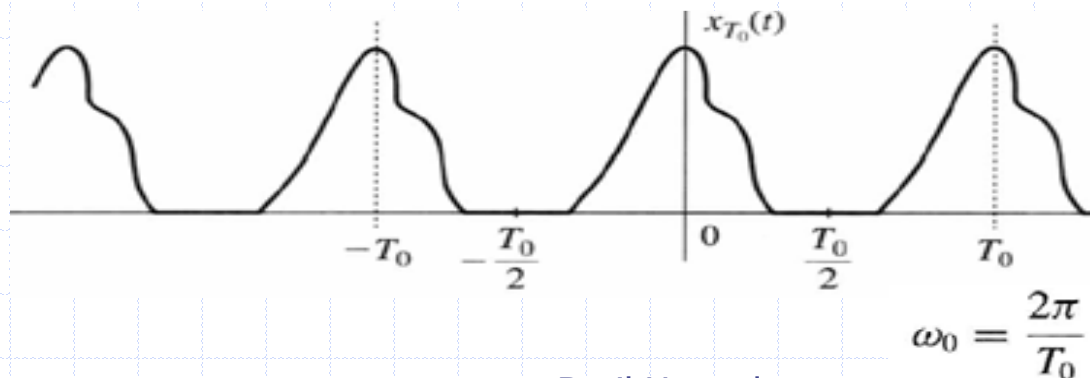
7.1 Aperiodic Signal Representation by Fourier Integral

The forward and inverse **Fourier Transform** are defined for **aperiodic** signal as:



$$X(\omega) = \mathcal{F}[x(t)] = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt$$
$$x(t) = \mathcal{F}^{-1}[X(\omega)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)e^{j\omega t} d\omega$$

Fourier series is used for **periodic** signals



$$x_{T_0}(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}$$
$$D_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x_{T_0}(t) e^{-jn\omega_0 t} dt$$

7.1 Aperiodic Signal Representation by Fourier Integral

Fourier Series: Used for periodic signals

Fourier Transform: Used for non-periodic signals (although we will see later that it can also be used for periodic signals)

If $X(\omega)$ is the Fourier transform of $x(t)$...then we can write this in several ways:

1. $x(t) \leftrightarrow X(\omega)$

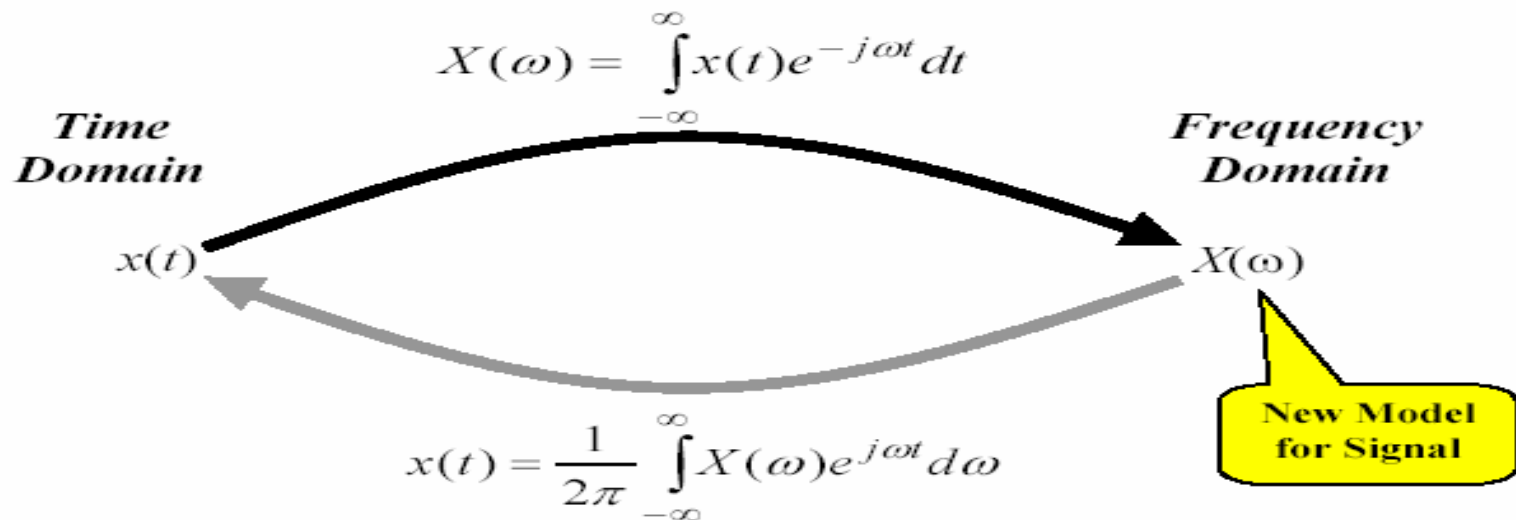
2. $X(\omega) = \mathcal{F}\{x(t)\} \Rightarrow \mathcal{F}\{ \}$ is an “operator” that operates on $x(t)$ to give $X(\omega)$

3. $x(t) = \mathcal{F}^{-1}\{X(\omega)\} \Rightarrow \mathcal{F}^{-1}\{ \}$ is an “operator” that operates on $X(\omega)$ to give $x(t)$

7.1 Aperiodic Signal Representation by Fourier Integral

Fourier Transform Viewpoint

View FT as a transformation into a new “domain”



$x(t)$ is the “time domain” description of the signal

$X(\omega)$ is the “frequency domain” description of the signal

Convergence (Existence) of Fourier transform; a function $f(t)$ has a Fourier transform if the integral converges $\int_{-\infty}^{\infty} |f(t)| dt < \infty$

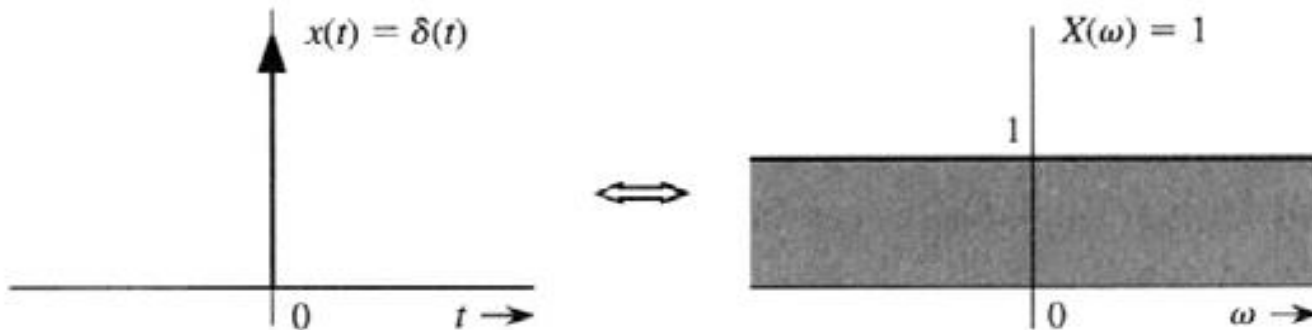
7.2 Transform of Some Useful functions

Fourier Transform of unit impulse $\delta(t)$

Using the sampling property of the impulse, we get:

$$\int_{t_0-\varepsilon}^{t_0+\varepsilon} f(t)\delta(t-t_0)dt = f(t_0) \quad \forall \varepsilon > 0$$

$$\mathcal{F}[\delta(t)] = \int_{-\infty}^{\infty} \delta(t)e^{-j\omega t}dt = 1 \quad \delta(t) \Longleftrightarrow 1$$



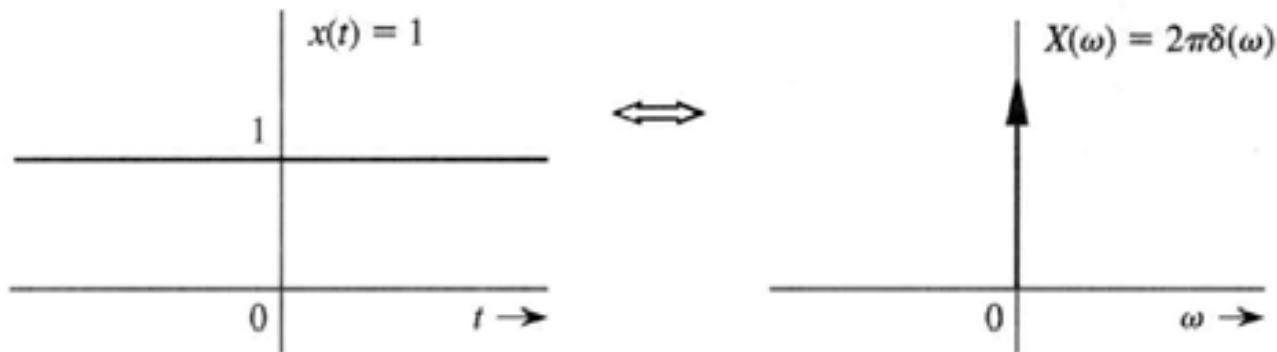
7.2 Transform of Some Useful functions

Inverse Fourier Transform of $\delta(\omega)$

Using the sampling property of the impulse, we get:

$$\mathcal{F}^{-1}[\delta(\omega)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\omega) e^{j\omega t} d\omega = \frac{1}{2\pi}$$

$$\frac{1}{2\pi} \Longleftrightarrow \delta(\omega) \quad \longrightarrow \quad 1 \Longleftrightarrow 2\pi \delta(\omega)$$



7.2 Transform of Some Useful functions

Fourier Transform Table

$x(t)$	$X(f)$	$X(\omega)$
$\delta(t)$	1	1
1	$\delta(f)$	$2\pi\delta(\omega)$
$\delta(t - t_0)$	$e^{-j2\pi ft_0}$	$e^{-j\omega t_0}$
$e^{j2\pi ft_0}$	$\delta(f - f_0)$	$2\pi\delta(\omega - \omega_0)$
$\cos(2\pi f_0 t)$	$\frac{1}{2}[\delta(f - f_0) + \delta(f + f_0)]$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$
$\sin(2\pi f_0 t)$	$\frac{1}{2j}[\delta(f - f_0) - \delta(f + f_0)]$	$-j\pi[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$
$\text{rect}(t)$	$\text{sinc}(f)$	$\text{sinc}\left(\frac{\omega}{2\pi}\right)$
$\text{sinc}(t)$	$\text{rect}(f)$	$\text{rect}\left(\frac{\omega}{2\pi}\right)$
$\Lambda(t)$	$\text{sinc}^2(f)$	$\text{sinc}^2\left(\frac{\omega}{2\pi}\right)$
$\text{sinc}^2(t)$	$\Lambda(f)$	$\Lambda\left(\frac{\omega}{2\pi}\right)$
$e^{-\alpha t}u(t), \alpha > 0$	$\frac{1}{\alpha + j2\pi f}$	$\frac{1}{\alpha + j\omega}$
$te^{-\alpha t}u(t), \alpha > 0$	$\frac{1}{(\alpha + j2\pi f)^2}$	$\frac{1}{(\alpha + j\omega)^2}$
$e^{-\alpha t }, \alpha > 0$	$\frac{2\alpha}{(\alpha^2 + (2\pi f)^2)}$	$\frac{2\alpha}{(\alpha^2 + (\omega)^2)}$
$e^{-\alpha t^2}$	$e^{-\alpha f^2}$	$e^{-\alpha \omega^2}$
$\text{sgn}(t)$	$\frac{1}{j\pi f}$	$\frac{2}{j\omega}$
$u(t)$	$\frac{1}{2}\delta(f) + \frac{1}{j2\pi f}$	$\pi\delta(\omega) + \frac{1}{j\omega}$
$\frac{d}{dt}\delta(t)$	$j2\pi f$	$j\omega$
$\sum_{n=-\infty}^{\infty} \delta(t - nT_0)$	$\frac{1}{T_0} \sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{T_0}\right)$	$\frac{1}{T_0} \sum_{n=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi n}{T_0}\right)$

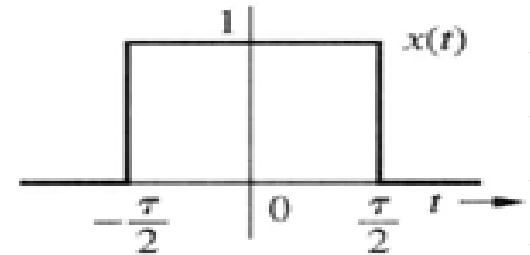
	Time Domain	Frequency Domain
	$x(t) = \mathcal{F}^{-1}\{X(\omega)\}$	$X(\omega) = \mathcal{F}\{x(t)\}$
1	$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt$	$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)e^{j\omega t} d\omega$
2	1	$2\pi\delta(\omega)$
3	$-0.5 + u(t)$	$\frac{1}{j\omega}$
4	$\delta(t)$	1
5	$\delta(t - c)$	$e^{-j\omega c}$
6	$u(t)$	$\pi\delta(\omega) + \frac{1}{j\omega}$
7	$e^{-bt}u(t) \ (b > 0)$	$\frac{1}{j\omega + b}$
8	$\cos \omega_0 t$	$\pi [\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]$
9	$\cos(\omega_0 t + \theta)$	$\pi [e^{-j\theta}\delta(\omega + \omega_0) + e^{j\theta}\delta(\omega - \omega_0)]$
10	$\sin \omega_0 t$	$j\pi [\delta(\omega + \omega_0) - \delta(\omega - \omega_0)]$
11	$\sin(\omega_0 t + \theta)$	$j\pi [e^{-j\theta}\delta(\omega + \omega_0) - e^{j\theta}\delta(\omega - \omega_0)]$
12	$\text{rect}\left(\frac{t}{\tau}\right)$	$\tau \text{sinc}\left(\frac{\tau\omega}{2\pi}\right)$

7.2 Transform of Some Useful functions

Fourier Transform of $x(t) = \text{rect}(t/\tau)$

Evaluation:

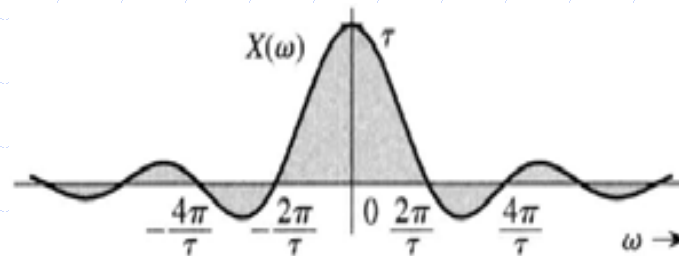
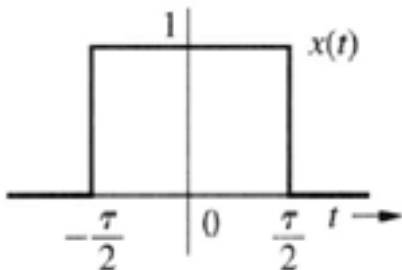
$$X(\omega) = \int_{-\infty}^{\infty} \text{rect}\left(\frac{t}{\tau}\right) e^{-j\omega t} dt$$



Since $\text{rect}(t/\tau) = 1$ for $-\tau/2 < t < \tau/2$ and 0 otherwise

$$X(\omega) = \int_{-\tau/2}^{\tau/2} e^{-j\omega t} dt = -\frac{1}{j\omega} (e^{-j\omega\tau/2} - e^{j\omega\tau/2}) = \frac{2 \sin\left(\frac{\omega\tau}{2}\right)}{\omega} = \tau \frac{\sin\left(\frac{\omega\tau}{2}\right)}{\left(\frac{\omega\tau}{2}\right)} = \tau \text{sinc}\left(\frac{\omega\tau}{2}\right)$$

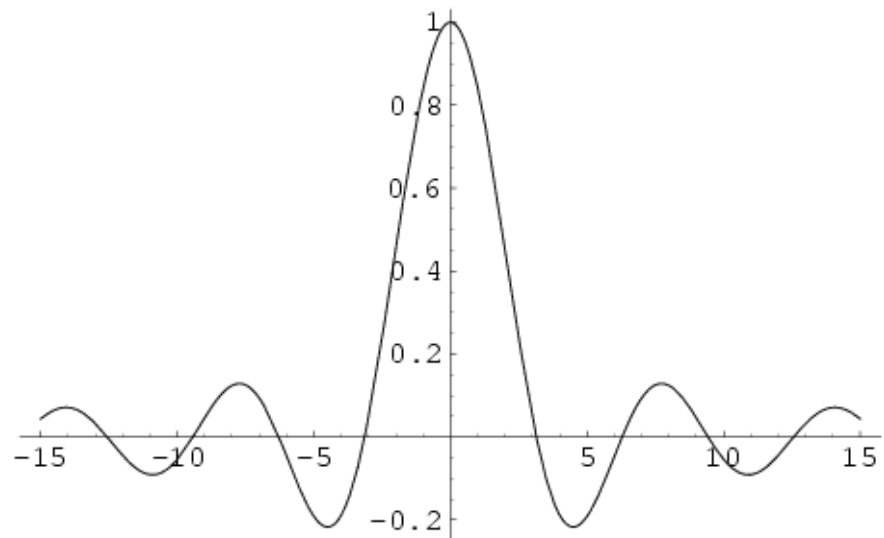
$$\text{rect}\left(\frac{t}{\tau}\right) \Longleftrightarrow \tau \text{sinc}\left(\frac{\omega\tau}{2}\right)$$



7.2 Transform of Some Useful functions

The **sinc function** is the Fourier transform of the rectangular function. It is used in the concept of reconstructing a continuous **bandlimited** signal from uniformly spaced samples of that signal.

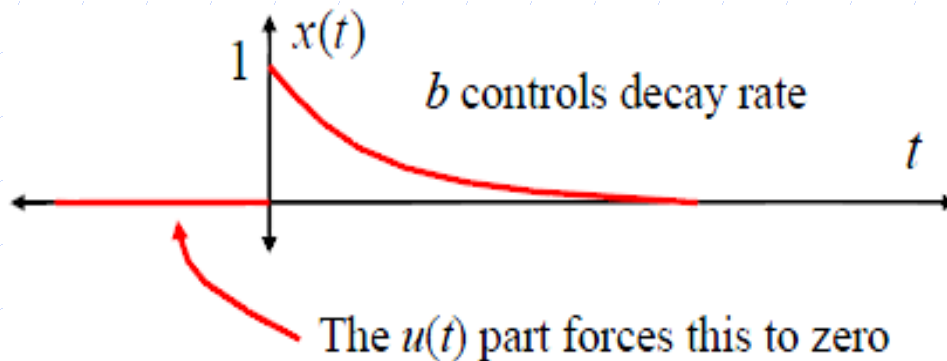
$$\text{sinc}(x) \equiv \begin{cases} 1 & \text{for } x = 0 \\ \frac{\sin x}{x} & \text{otherwise,} \end{cases}$$



7.2 Transform of Some Useful functions

Example: Given a signal $x(t) = e^{-bt}u(t)$ find $X(\omega)$ if $b > 0$

Solution: First see what $x(t)$ looks like:



Now...apply the definition of the Fourier transform. Recall the general form:

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt$$

7.2 Transform of Some Useful functions

Now plug in for our signal:

$$X(\omega) = \int_{-\infty}^{\infty} \underbrace{e^{-bt} u(t) e^{-j\omega t}}_{\text{integrand} = 0 \text{ for } t < 0 \text{ due to the } u(t)} dt = \int_0^{\infty} e^{-bt} e^{-j\omega t} dt = \int_0^{\infty} e^{-(b+j\omega)t} dt$$

Set lower limit to 0
and then $u(t) = 1$ over
integration range

$$= \left[\frac{-1}{b+j\omega} e^{-(b+j\omega)t} \right]_{t=0}^{t=\infty} = \frac{-1}{b+j\omega} \left[e^{-(b+j\omega)\infty} - e^{-(b+j\omega)0} \right]$$

$$= \frac{-1}{b+j\omega} \left[\underbrace{e^{-b\infty}}_{=0} \underbrace{e^{-j\omega\infty}}_{\text{mag}=1} - \underbrace{e^0}_{=1} \right] = \frac{-1}{b+j\omega} [0 - 1] = \frac{1}{b+j\omega}$$

7.2 Transform of Some Useful functions

Inverse Fourier Transform of $\delta(\omega - \omega_0)$

Using the sampling property of the impulse, we get:

$$\mathcal{F}^{-1}[\delta(\omega - \omega_0)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\omega - \omega_0) e^{j\omega t} d\omega = \frac{1}{2\pi} e^{j\omega_0 t}$$

Spectrum of an everlasting exponential $e^{j\omega_0 t}$ is a single impulse at $\omega = \omega_0$.

$$\frac{1}{2\pi} e^{j\omega_0 t} \Longleftrightarrow \delta(\omega - \omega_0)$$

or

$$e^{j\omega_0 t} \Longleftrightarrow 2\pi \delta(\omega - \omega_0)$$

and

$$e^{-j\omega_0 t} \Longleftrightarrow 2\pi \delta(\omega + \omega_0)$$

7.2 Transform of Some Useful functions

Fourier Transform of everlasting sinusoid $\cos \omega_0 t$

Remember Euler formula: $\cos \omega_0 t = \frac{1}{2}(e^{j\omega_0 t} + e^{-j\omega_0 t})$

Use results from previous slide, we get:

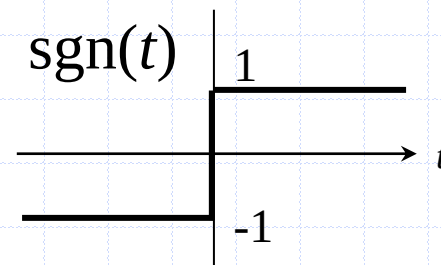
$$\cos \omega_0 t \Longleftrightarrow \pi[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]$$

7.2 Transform of Some Useful functions

Ex. Find Fourier Transform of shown Fig.

Solution:

$$\text{sgn}(t) = \begin{cases} 1 & t > 0 \\ 0 & t = 0 \\ -1 & t < 0 \end{cases}$$



$$\text{sgn}(t) = u(t) - u(-t) = \lim_{a \rightarrow 0} [e^{-at}u(t) - e^{at}u(-t)]$$

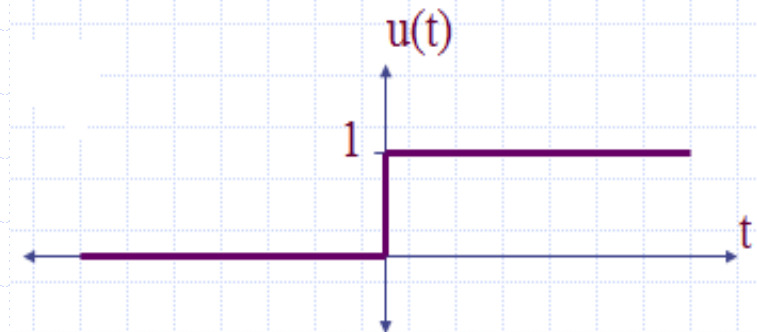
$$F\{\text{sgn}(t)\} = \lim_{a \rightarrow 0} [F[e^{-at}u(t)] - F[e^{at}u(-t)]]$$

$$= \lim_{a \rightarrow 0} \left[\frac{1}{a + j\omega} - \frac{1}{a - j\omega} \right] = \lim_{a \rightarrow 0} \left[\frac{-2j\omega}{a^2 + \omega^2} \right] = \frac{2}{j\omega}$$

7.2 Transform of Some Useful functions

Find Fourier transform Unit Step

$$u(t) = \begin{cases} 1 & , t > 0 \\ 0 & , t < 0 \end{cases}$$



Solution:

$$u(t) = \frac{1}{2} + \frac{1}{2} \text{Sgn}(t) \quad \text{take Fourier transform}$$

$$F(u(t)) = F\left\{\frac{1}{2}\right\} + F\left\{\frac{1}{2} \text{Sgn}(t)\right\} = \frac{1}{2}(2\pi\delta(\omega)) + \frac{1}{2}\left(\frac{2}{j\omega}\right)$$

$$U(\omega) = \pi\delta(\omega) + \frac{1}{j\omega}$$

7.3 Fourier Transform Properties

As we have seen, finding the FT can be tedious(it can even be difficult)

But...there are certain properties that can often make things easier.

Also, these properties can sometimes be the key to understanding how the FT can be used in a given application.

So...even though these results may at first seem like “just boring math” they are important tools that let signal processing engineers understand how to build things like cell phones, radars, mp3 processing, etc.

7.3 Fourier Transform Properties

Linearity

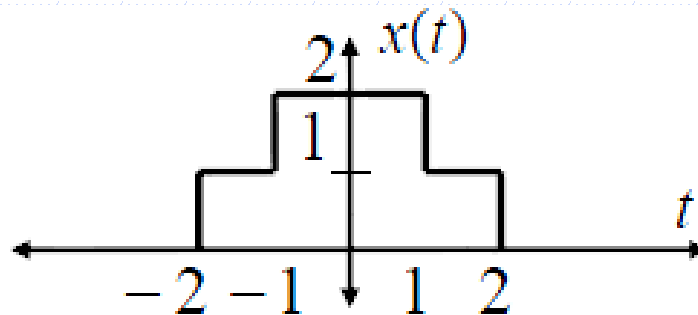
If $x_1(t) \Longleftrightarrow X_1(\omega)$ and $x_2(t) \Longleftrightarrow X_2(\omega)$

Then

$$a_1 x_1(t) + a_2 x_2(t) \Longleftrightarrow a_1 X_1(\omega) + a_2 X_2(\omega)$$

Example Application of “Linearity of FT”:

Suppose we need to find the FT of the following signal...



7.3 Fourier Transform Properties

Solution:

Finding this using straight-forward application of the definition of FT is not difficult but it is tedious:

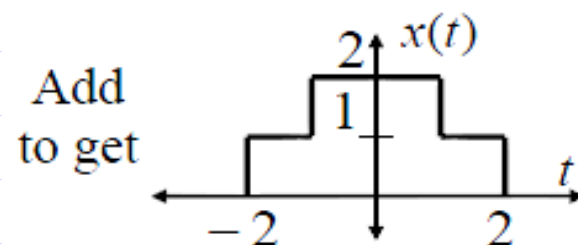
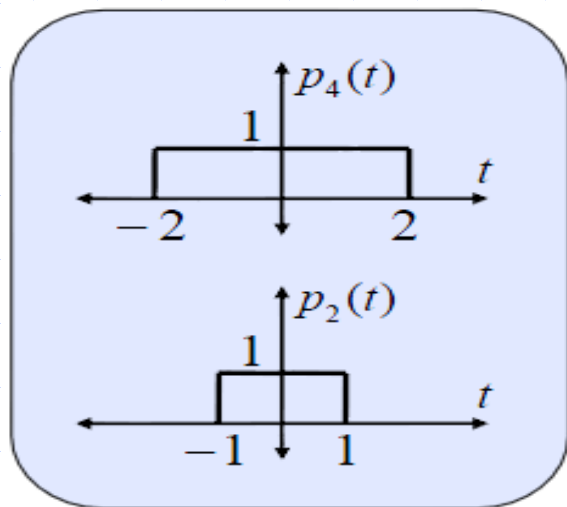
$$\mathcal{F}\{x(t)\} = \int_{-2}^{-1} e^{-j\omega t} dt + 2 \int_{-1}^1 e^{-j\omega t} dt + \int_1^2 e^{-j\omega t} dt$$

So...we look for short-cuts:

- One way is to recognize that each of these integrals is basically the same
- Another way is to break $x(t)$ down into a sum of signals on our table!!!

7.3 Fourier Transform Properties

Break a complicated signal down into simple signals before finding FT:



Mathematically we write: $x(t) = p_4(t) + p_2(t) \quad \rightarrow \quad X(\omega) = P_4(\omega) + P_2(\omega)$

From FT Table we have a known result for the FT of a pulse, so...

$$X(\omega) = 4 \operatorname{sinc}\left(\frac{2\omega}{\pi}\right) + 2 \operatorname{sinc}\left(\frac{\omega}{\pi}\right)$$

7.3 Fourier Transform Properties

Duality

If $x(t) \Longleftrightarrow X(\omega)$ then $X(t) \Longleftrightarrow 2\pi x(-\omega)$

Proof: From definition of inverse FT (previous slide), we get

Hence
$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(u) e^{jut} du$$

$$2\pi x(-t) = \int_{-\infty}^{\infty} X(u) e^{-jut} du$$

Change **t** to **ω** yield, and use definition of FT, we get:

$$2\pi x(-\omega) \Longleftrightarrow X(t)$$

7.3 Fourier Transform Properties

Duality

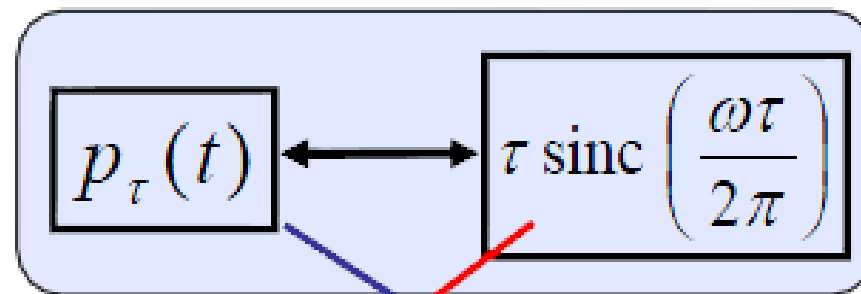
$$\begin{array}{ccc} & X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt & \\ x(t) & \xrightarrow{\hspace{10em}} & X(\omega) \\ & x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega & \end{array}$$

Suppose we have a FT table that a FT **Pair A**...we can get the dual **Pair B** using the general Duality Property:

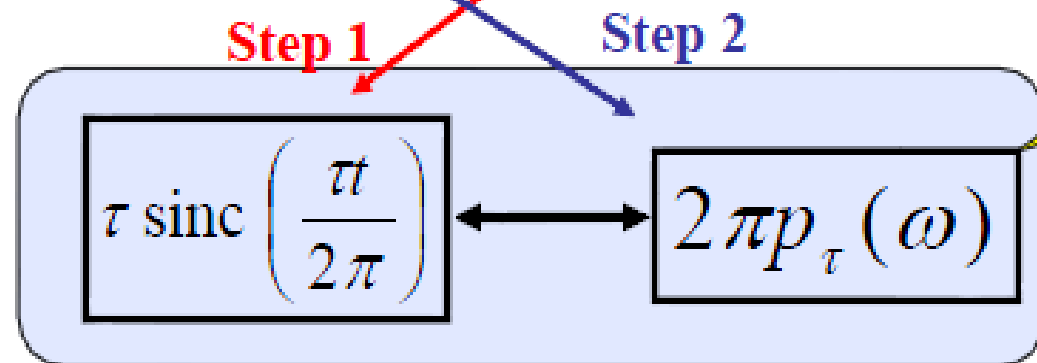
1. Take the FT side of (known) **Pair A** and replace ω by t and move it to the time-domain side of the table of the (unknown) **Pair B**.
2. Take the time-domain side of the (known) **Pair A** and replace t by $-\omega$, multiply by 2π , and then move it to the FT side of the table of the (unknown) **Pair B**.

7.3 Fourier Transform Properties

Pair A



Pair B

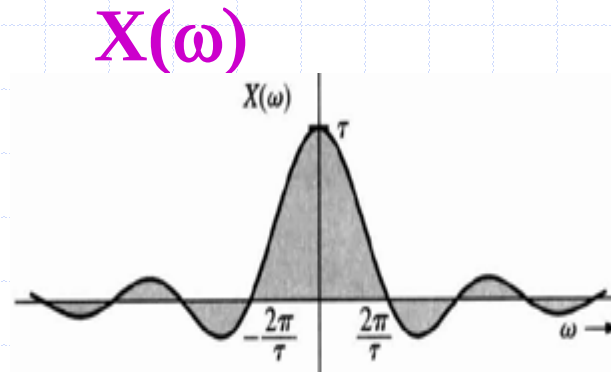
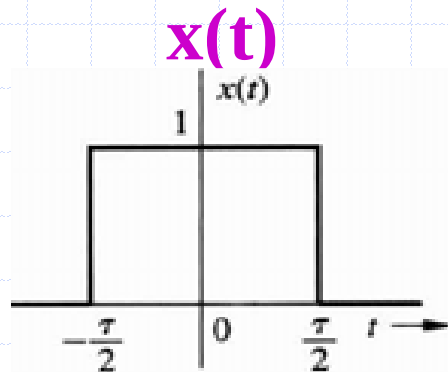


7.3 Fourier Transform Properties

Example: Find Fourier transform of $\tau \text{sinc}(t \tau / 2)$

Solution: we have from FT Table;

$$\text{rect}(t/\tau) \longleftrightarrow \tau \text{sinc}(\omega \tau / 2)$$

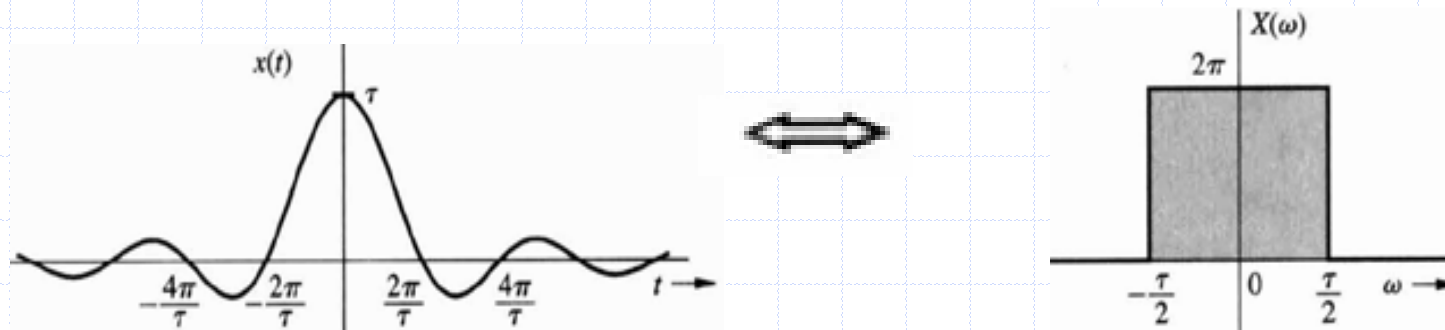


Change $\omega \longrightarrow t$

7.3 Fourier Transform Properties

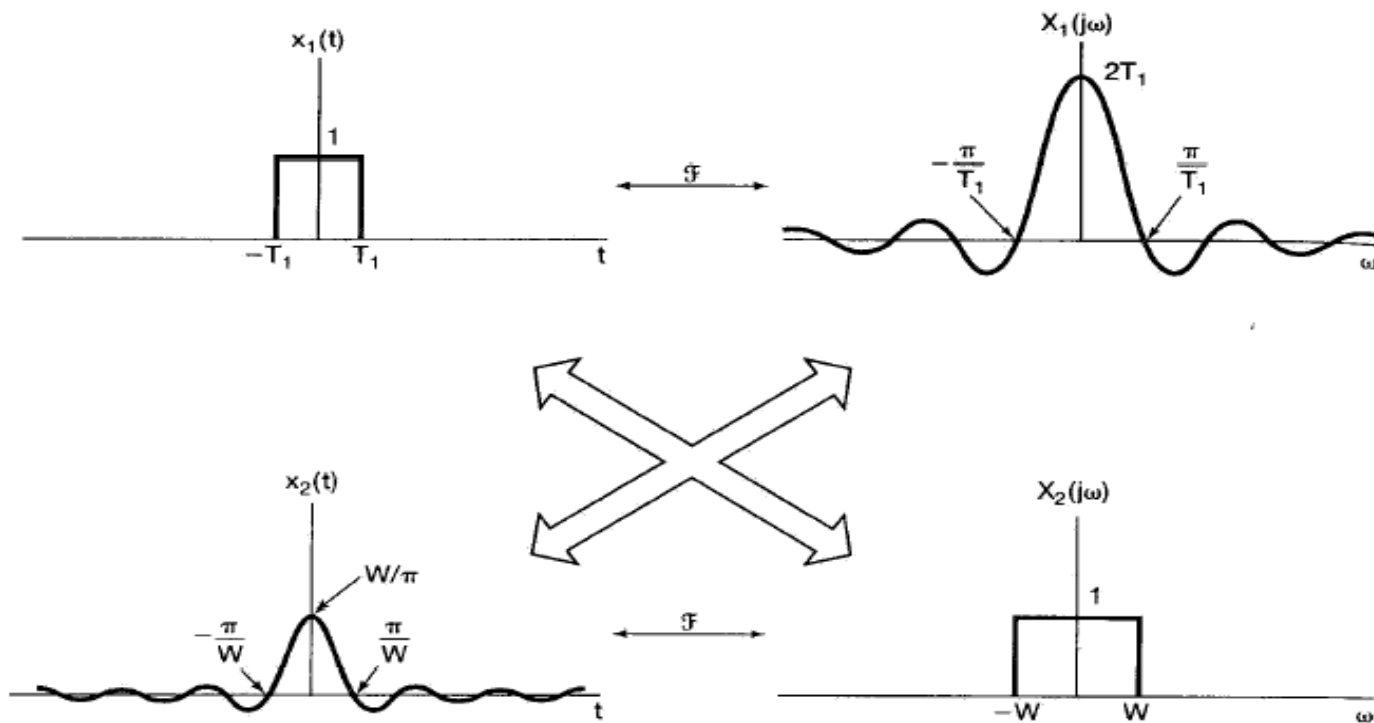
$$\underbrace{\tau \operatorname{sinc}\left(\frac{\tau t}{2}\right)}_{X(t)} \iff \underbrace{2\pi \operatorname{rect}\left(\frac{-\omega}{\tau}\right)}_{2\pi x(-\omega)} = 2\pi \operatorname{rect}\left(\frac{\omega}{\tau}\right)$$

$\operatorname{rect}(-x) = \operatorname{rect}(x)$ because rect is even function



7.3 Fourier Transform Properties

Duality Property



7.3 Fourier Transform Properties

Time-Shifting

If $x(t) \Longleftrightarrow X(\omega)$

Then

$$x(t - t_0) \Longleftrightarrow X(\omega)e^{-j\omega t_0}$$

Similarly

$$x(t)e^{j\omega_0 t} \longleftrightarrow X(\omega - \omega_0)$$

Freq Shift

Ex. Find $F\{x(t+1)\}$ given $X(\omega) = \text{rect}[(\omega-1)/2]$

Solution: $F\{x(t+1)\} = e^{j\omega} X(\omega) = e^{j\omega} \text{rect}\frac{(\omega-1)}{2}$

7.3 Fourier Transform Properties

Time Scaling

If $x(t) \Longleftrightarrow X(\omega)$

then for any real constant a ,

$$x(at) \Longleftrightarrow \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$$

Ex. Find $F\{\alpha \operatorname{rect}\left(\alpha \frac{t}{\tau}\right)\}$

Solution: From FT Table $F\left\{\operatorname{rect}\left(\frac{t}{\tau}\right)\right\} = \tau \operatorname{Sinc} \frac{\omega\tau}{2\pi}$

$$F\left\{\alpha \operatorname{rect}\left(\alpha \frac{t}{\tau}\right)\right\} = \frac{\alpha}{|\alpha|} \tau \operatorname{Sinc} \frac{\frac{\omega}{\alpha} \tau}{2\pi} = \tau \operatorname{Sinc} \frac{\omega\tau}{2\pi\alpha}$$

7.3 Fourier Transform Properties

Ex. Find $F[x(-2t + 4)]$ given $X(\omega) = \text{rect}[(\omega - 1)/2]$

Solution: $F[x(-2t + 4)] = F[x(-2(t - 2))]$

$$= \frac{1}{|-2|} X\left(\frac{\omega}{-2}\right) e^{-j2\omega} = \frac{e^{-j2\omega}}{2} \text{rect}\left(\frac{\omega + 2}{4}\right)$$

7.3 Fourier Transform Properties

Differentiation:

$$\text{If } x(t) \Longleftrightarrow X(\omega) \quad \text{Then } \frac{dx}{dt} \Longleftrightarrow j\omega X(\omega)$$

$$\frac{d^n x}{dt^n} \Longleftrightarrow (j\omega)^n X(\omega)$$

Ex. Find $F\{dx(t)/dt\}$ given $X(\omega) = \text{rect}[(\omega-1)/2]$

Solution:

$$F\{dx(t)/dt\} = j\omega X(\omega) = j\omega \text{rect}[(\omega-1)/2]$$

7.3 Fourier Transform Properties

Multiplication by a Power of t

If $x(t) \longleftrightarrow X(\omega)$ Then $t^n x(t) \longleftrightarrow j^n \frac{d^n}{d\omega^n} X(\omega)$

n: Positive Integer

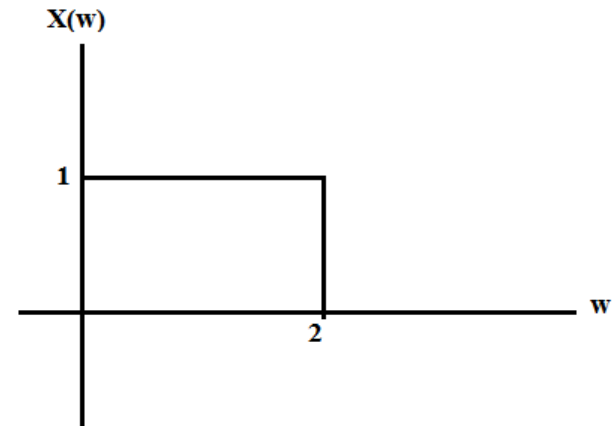
Ex. Find $F\{t x(t)\}$ given $X(\omega) = \text{rect}[(\omega-1)/2]$

Solution:

$$F\{t x(t)\} = j \frac{d}{d\omega} X(\omega)$$

$$X(\omega) = U(\omega) - U(\omega-2)$$

$$F\{t x(t)\} = j[\delta(\omega) - \delta(\omega - 2)]$$



7.3 Fourier Transform Properties

Convolution

If $x_1(t) \Longleftrightarrow X_1(\omega)$ and $x_2(t) \Longleftrightarrow X_2(\omega)$

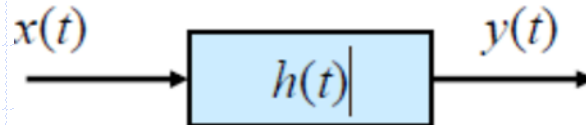
Then

$$x_1(t) * x_2(t) \Longleftrightarrow X_1(\omega) X_2(\omega) \quad (\text{time convolution})$$

$$x_1(t)x_2(t) \Longleftrightarrow \frac{1}{2\pi} X_1(\omega) * X_2(\omega) \quad (\text{frequency convolution})$$

Let $H(\omega)$ be the Fourier transform of the unit impulse response $h(t)$

$$h(t) \longleftrightarrow H(\omega)$$



Applying the time-convolution property to $y(t) = x(t) * h(t)$, we get:
 $Y(\omega) = X(\omega) H(\omega)$

7.3 Fourier Transform Properties

Convolution

$$x(t)y(t) \leftrightarrow \frac{1}{2\pi} X(\omega) * Y(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\lambda) Y(\omega - \lambda) d\lambda$$

This is the “dual” of the convolution property!!!

“Convolution in the
Time-Domain”

gives

“Multiplication in the
Frequency-Domain”

“Multiplication in
the Time-Domain”

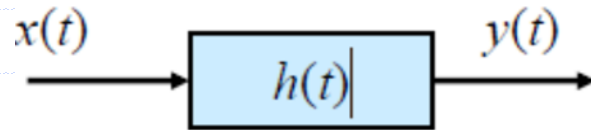
gives

“Convolution in the
Frequency-Domain”

7.3 Fourier Transform Properties

Ex. Given $h(t) = e^{-at}u(t)$ $x(t) = e^{-bt}u(t)$ $a > 0$

Find $y(t)$



Solution:

$y(t) = x(t) * h(t)$ using FT $Y(\omega) = X(\omega) H(\omega)$

From FT Table

$$X(\omega) = 1/a + j\omega ; \quad H(\omega) = 1/b + j\omega$$

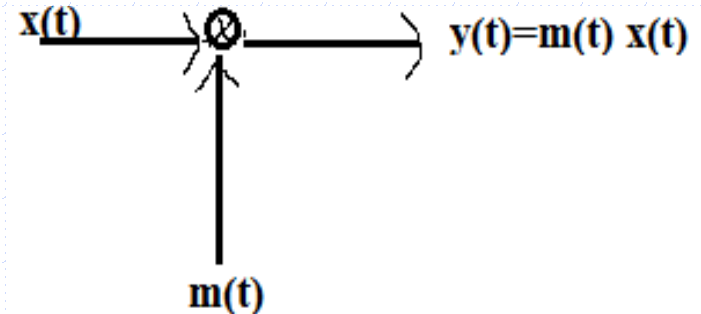
$$Y(\omega) = X(\omega) H(\omega) = (1/a + j\omega)(1/b + j\omega) = \frac{k_1}{a + j\omega} + \frac{k_2}{b + j\omega}$$

$$Y(\omega) = \frac{1}{a + j\omega} + \frac{1}{b + j\omega} \longrightarrow y(t) = \frac{1}{a - b} [e^{-bt} - e^{-at}]u(t)$$

7.3 Fourier Transform Properties

Modulation:

$$\begin{aligned} \text{If } x(t) &\longleftrightarrow X(\omega) \\ m(t) &\longleftrightarrow M(\omega) \end{aligned}$$



$$\text{Then } x(t) m(t) \longleftrightarrow \frac{1}{2\pi} [X(\omega) * M(\omega)]$$

7.7 Application of The Fourier Transform

Fourier transform, are tools that find extensive application in communication systems, signal processing, control systems, and many other varieties of engineering areas (such as):

- Circuit Analysis
- Amplitude Modulation
- Sampling Theorem
- Frequency multiplexing

7.7 Application of The Fourier Transform

Circuit Analysis by using FT

Ex Find $i(t)$ given $v(t) = 10e^{-t}u(t)$

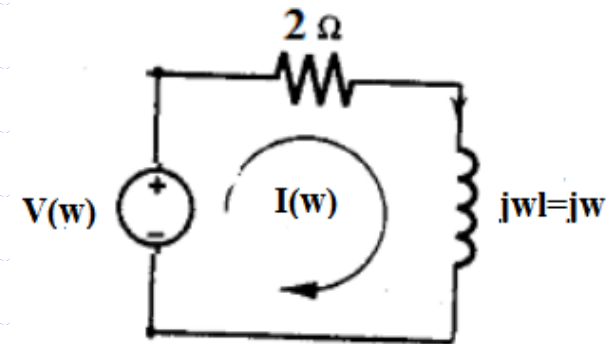
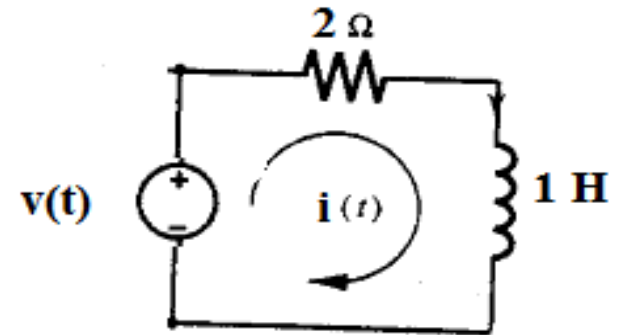
Solution:

Transform the ckt to FT.

$$V(\omega) = \frac{10}{1+j\omega}, \quad I(\omega) = \frac{V(\omega)}{Z}; \quad Z=2+j\omega$$

$$I(\omega) = \frac{\frac{10}{1+j\omega}}{2+j\omega} = \frac{k_1}{2+j\omega} + \frac{k_2}{1+j\omega} = \frac{-10}{2+j\omega} + \frac{10}{1+j\omega}$$

$$i(t) = [-10e^{-2t} + 10e^{-t}]u(t)$$



7.7 Application of The Fourier Transform

Amplitude Modulation (AM):

The goal of all communication system is to convey information from one point to another.

Prior to sending the information through the transmission channel the signal is converted to a useful form through what is known modulation.

Reasons for employing this type of conversion:

1. To transmit information efficiently.
2. To overcome hardware limitations.
3. To reduce noise and interference.

7.7 Application of The Fourier Transform

Essence of Amplitude Modulation (AM)

- For a transmission environment that only works at certain frequencies, people shift the input signal by multiplying them with either a complex exponential or by a sinusoidal signal.
- Multiplication done at the input end is called “modulation”.
- Multiplication done at the output end is called “demodulation”.

AM and FM Radio Frequencies

AM radio carrier frequencies are in the frequency range **535-1605 kHz**. Carrier frequencies of 540 to 1600 kHz are assigned at 10 kHz intervals.

FM radio band is from **88 to 108 MHz** between VHF television Channels 6 and 7. The FM stations are assigned center frequencies at 200 kHz separation starting at 88.1 MHz, for a maximum of 100 stations. These FM stations have a 75 kHz maximum deviation from the center frequency, which leaves 25 kHz upper and lower "guard bands" to minimize interaction with the adjacent frequency band.

AM and FM Radio Frequencies

10 kHz bandwidth from
540-1600 kHz for
106 possible bands

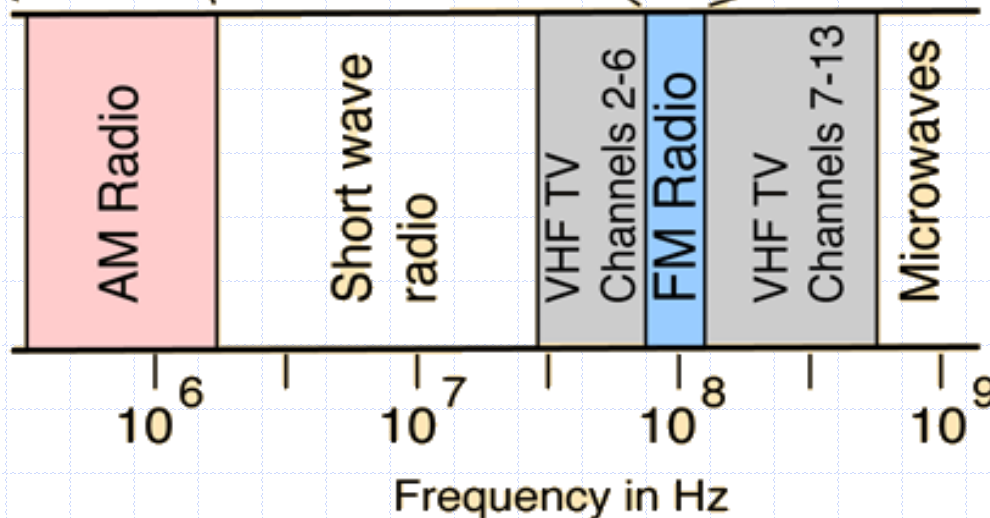


AM

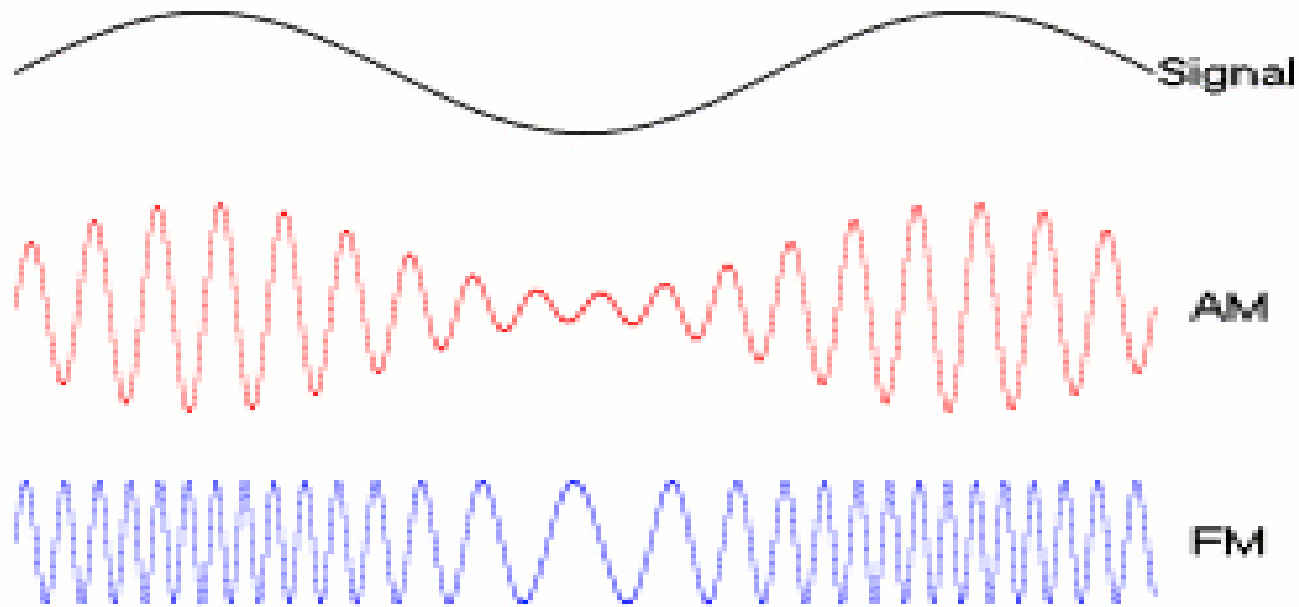
200 kHz bandwidth from
88.1-108.1 MHz for
100 possible bands



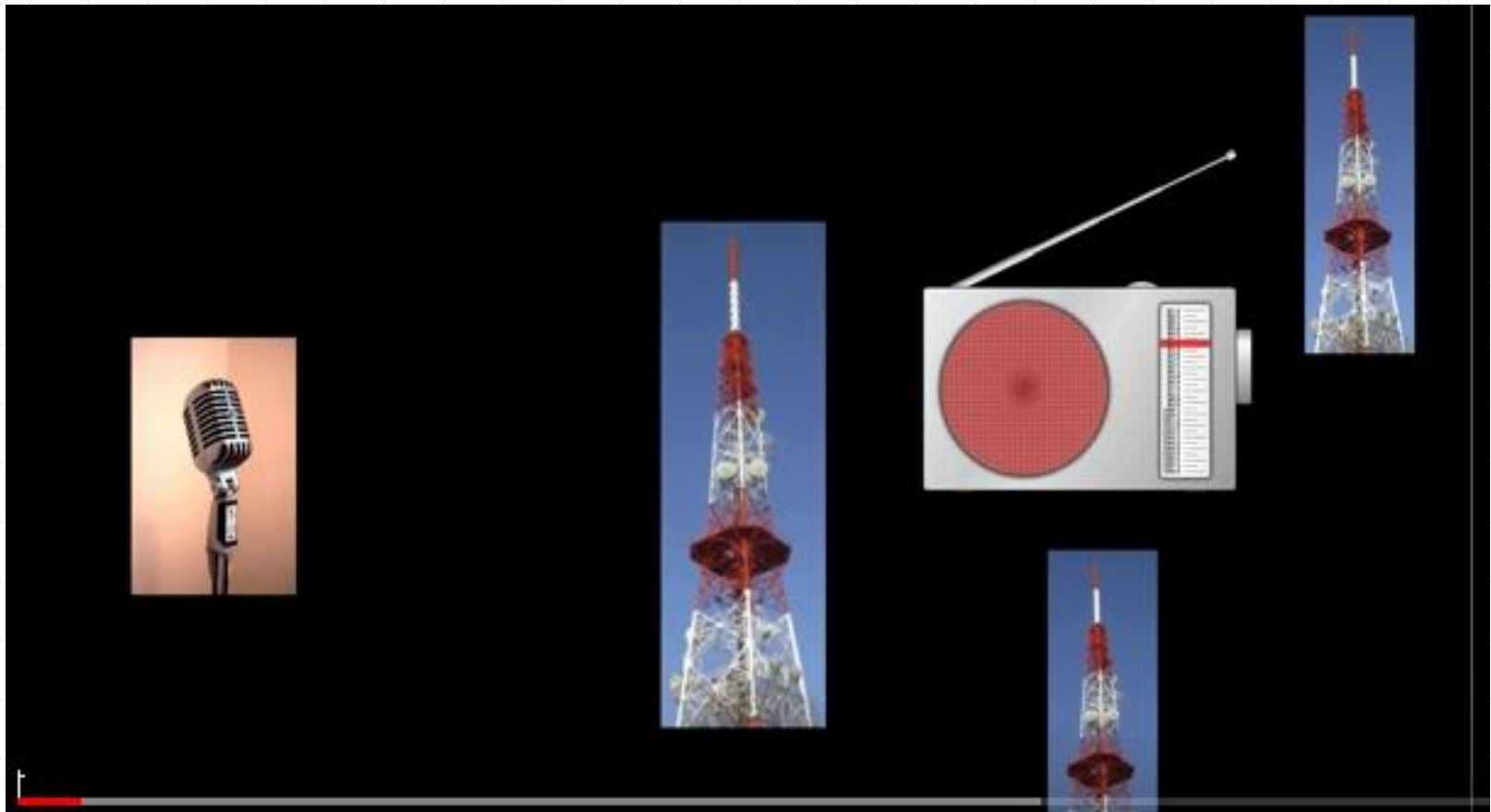
FM



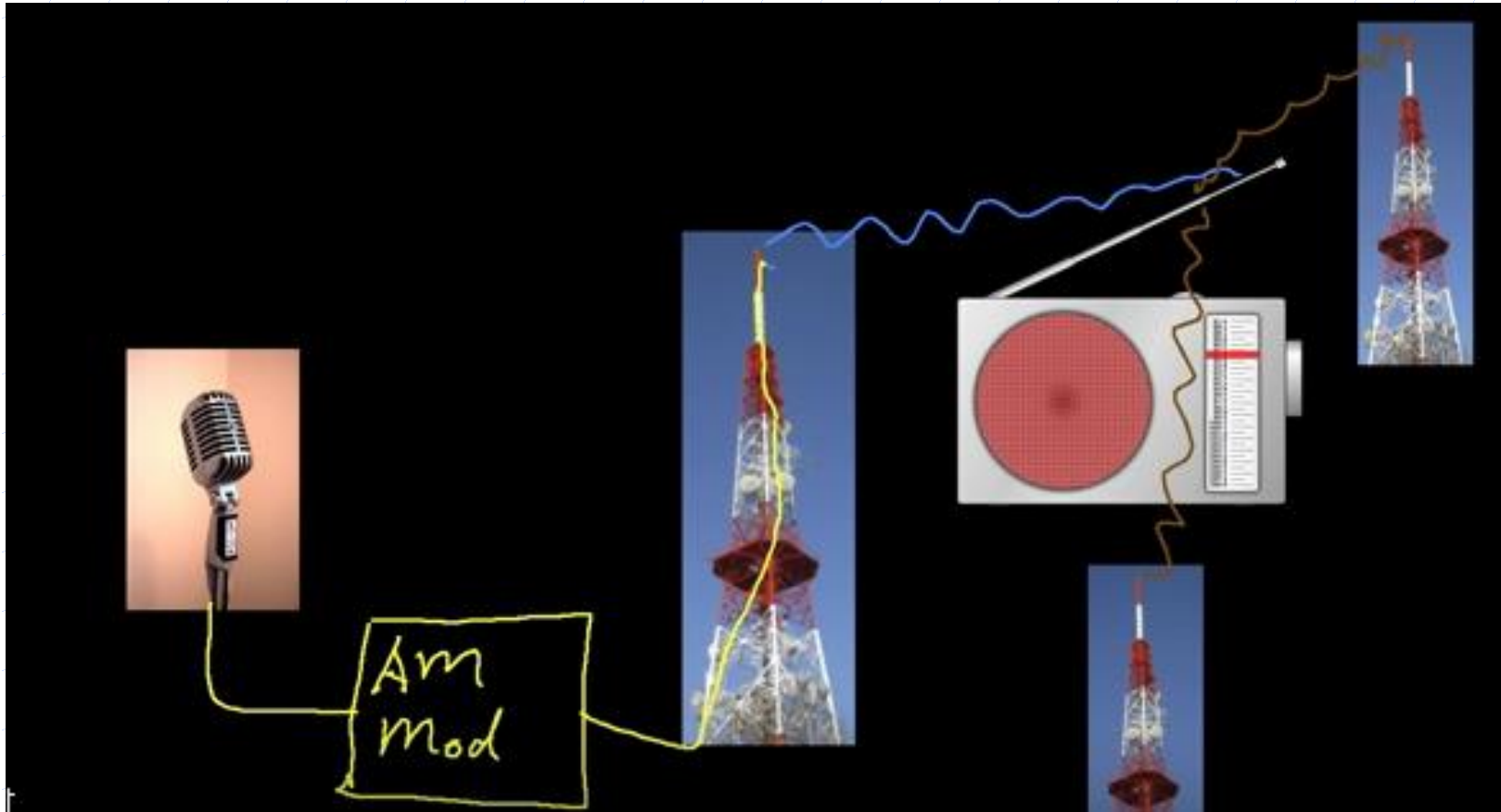
AM and FM Radio Frequencies



7.7 Application of The Fourier Transform

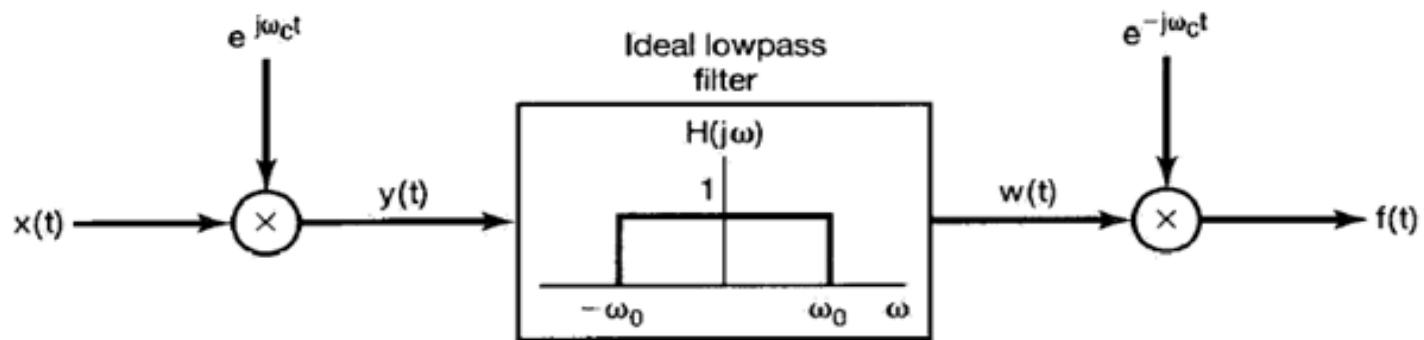


7.7 Application of The Fourier Transform



7.7 Application of The Fourier Transform

Example: Modulation and Demodulation in Signal Processing



$$y(t) = e^{j\omega_c t} x(t)$$

$$Y(j\omega) = X(j(\omega - \omega_c))$$

$$f(t) = e^{-j\omega_c t} w(t)$$

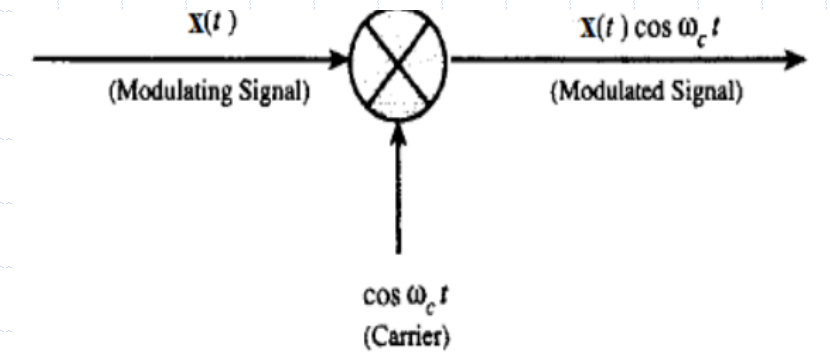
$$F(j\omega) = W(j(\omega + \omega_c))$$

7.7 Application of The Fourier Transform

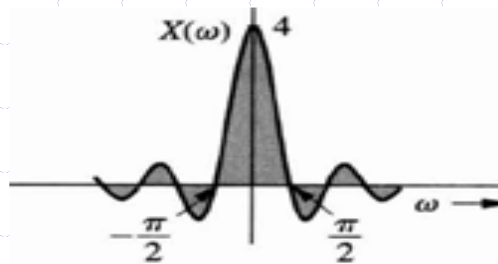
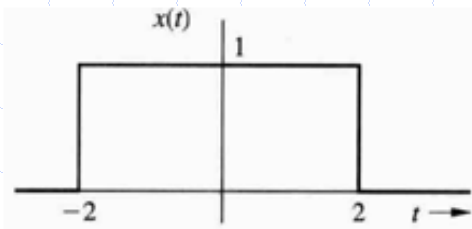
Ex. 7.15 P 710

Find and sketch the Fourier transform of the signal

Where $x(t) \cos 10t$ $x(t) = \text{rect}(t / 4)$.

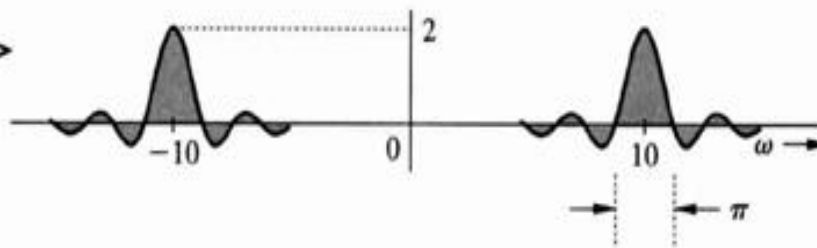
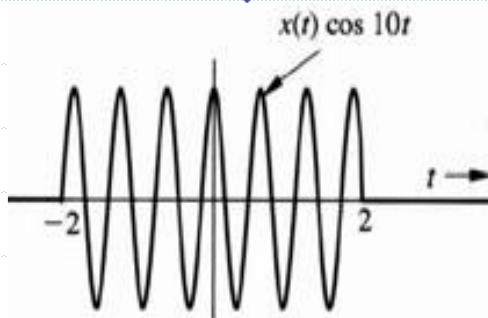
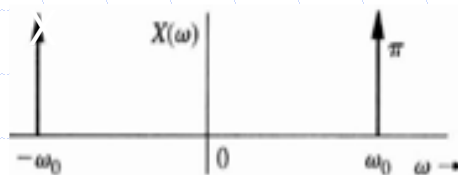
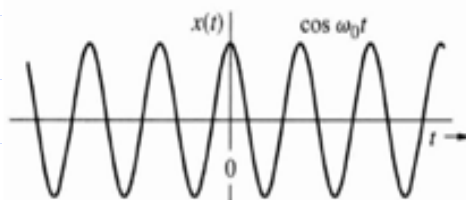


7.7 Application of The Fourier Transform



X

*



7.7 Application of The Fourier Transform

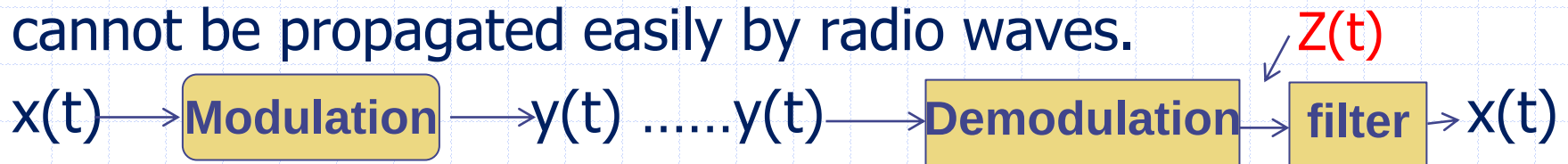
The output of the multiplier is:

$$y(t) = x(t) \cos 10 t$$

$$X(t) \cos 10 t \longleftrightarrow \frac{1}{2\pi} X(\omega) * \pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$$

$$Y(\omega) = \frac{1}{2} X(\omega) * [\delta(\omega - 10) + \delta(\omega + 10)]$$

The above process of shifting the spectrum of the signal by ($\omega_0=10$) is necessary because low- freq information signals cannot be propagated easily by radio waves.



- Demodulation shifts back the message spectrum to its original low frequency location. Demodulation is modulation followed by lowpass filtering

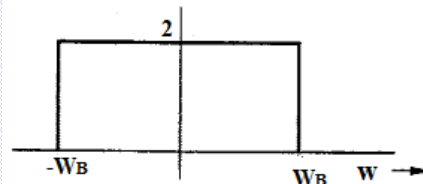
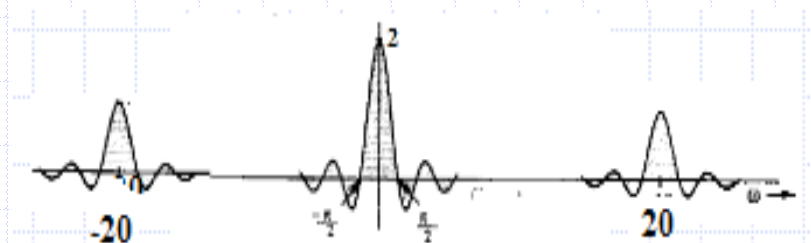
7.7 Application of The Fourier Transform

- Now we describe one technique for demodulation AM signal known as coherent demodulation.

$$z(t) = y(t) \cos \omega_0 t$$

$$Z(\omega) = \frac{1}{2} [Y(\omega - \omega_0) + Y(\omega + \omega_0)] = \frac{1}{2} \left\{ \frac{1}{2} [X(\omega - \omega_0 - \omega_0) + X(\omega - \omega_0 + \omega_0)] + \frac{1}{2} [X(\omega + \omega_0 + \omega_0) + X(\omega + \omega_0 - \omega_0)] \right\}$$

$$Z(\omega) = \frac{1}{4} X(\omega - 20) + \frac{1}{2} X(\omega) + \frac{1}{4} X(\omega + 20)$$

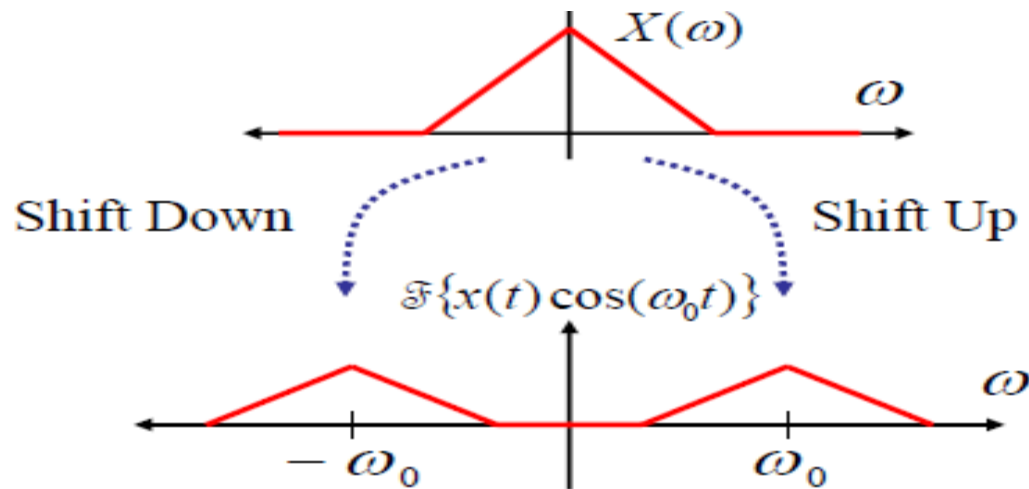


Low-Pass Filter

7.7 Application of The Fourier Transform

Visualizing the Result

$$x(t) \cos(\omega_0 t) \leftrightarrow \frac{1}{2} \left[\underbrace{X(\omega - \omega_0)}_{\text{Shift up}} + \underbrace{X(\omega + \omega_0)}_{\text{Shift down}} \right]$$

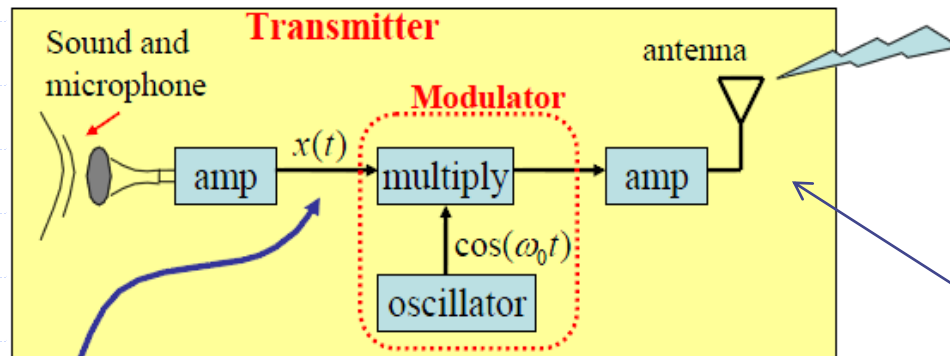


Interesting... This tells us how to move a signal's spectrum up to higher frequencies without changing the shape of the spectrum!!! What is that good for??? Well... only high frequencies will radiate from an antenna and propagate as electromagnetic waves and then induce a signal in a receiving antenna....

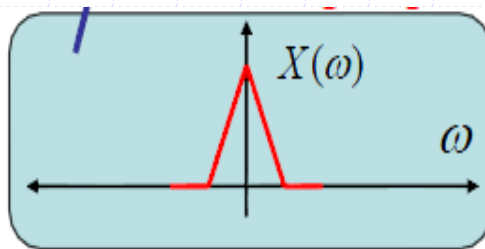
7.7 Application of The Fourier Transform

Application of Modulation Property to Radio Communication

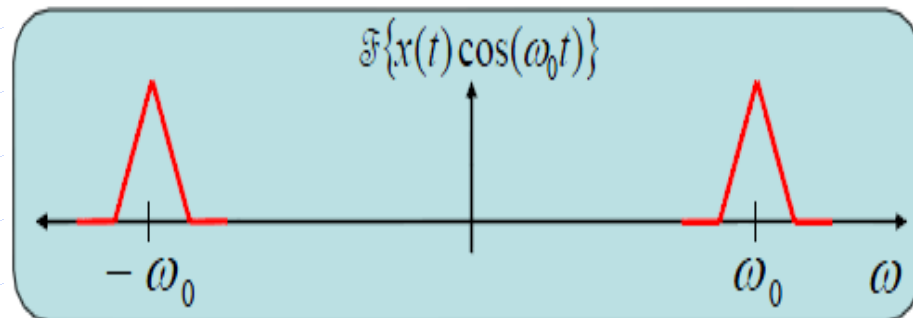
FT theory tells us what we need to do to make a **simple** radio system...
then electronics can be built to perform the operations that the FT theory:



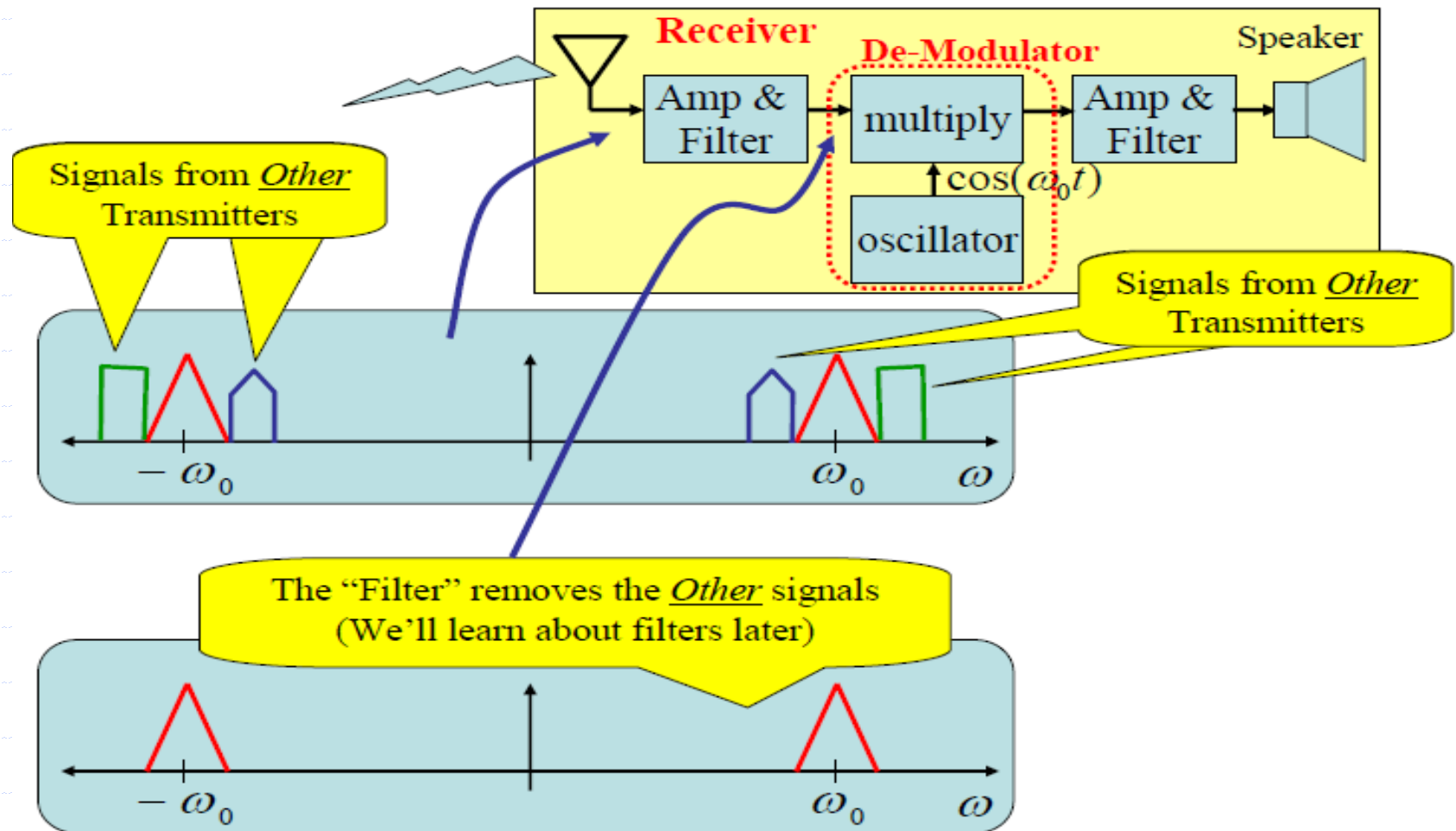
AM Radio: around 1 MHz
FM Radio: around 100 MHz
Cell Phones: around 900 MHz, around 1.8 GHz, around 1.9 GHz



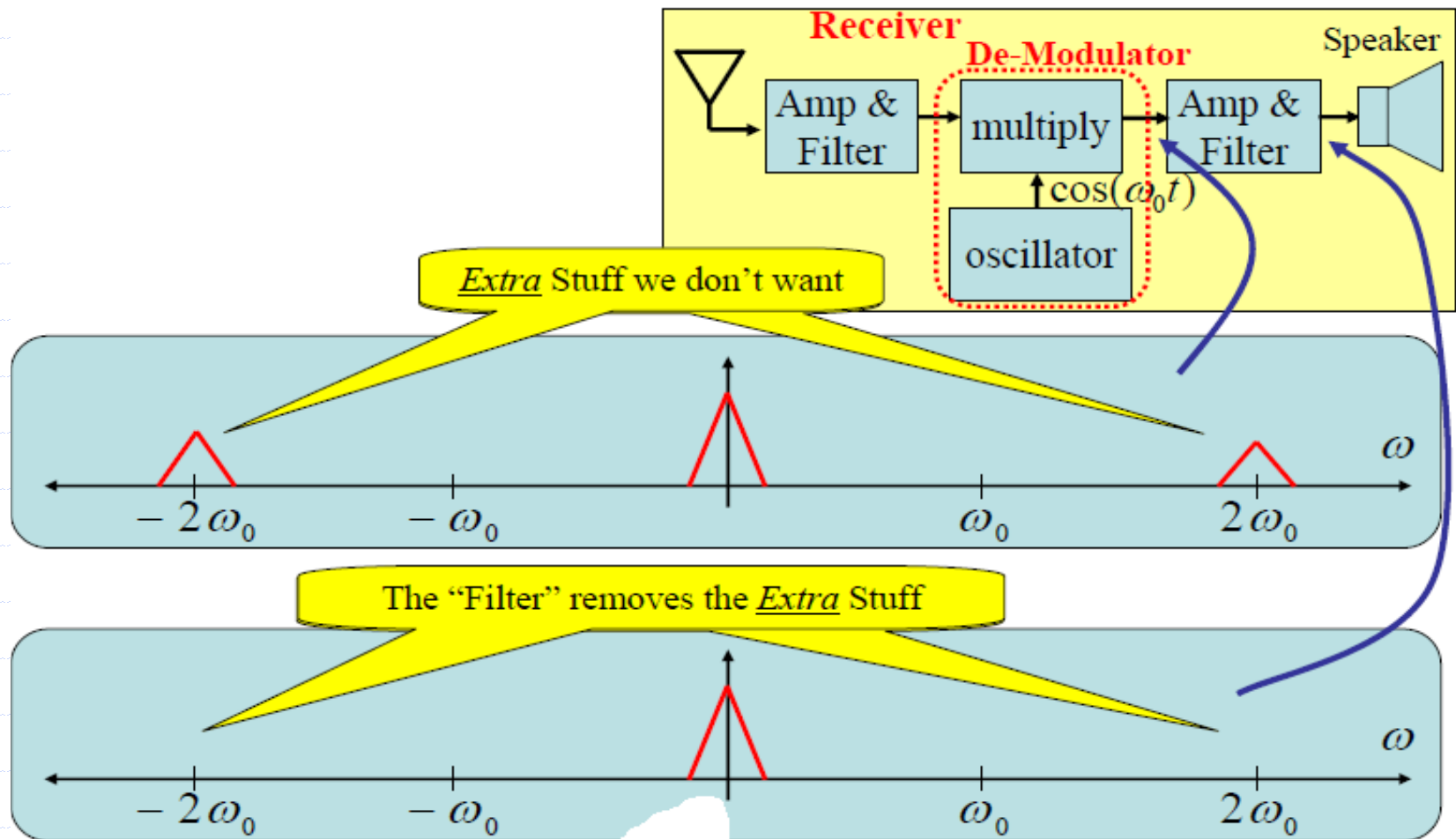
FT of Message Signal



7.7 Application of The Fourier Transform



7.7 Application of The Fourier Transform

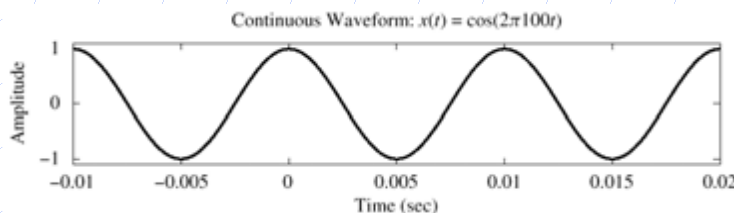
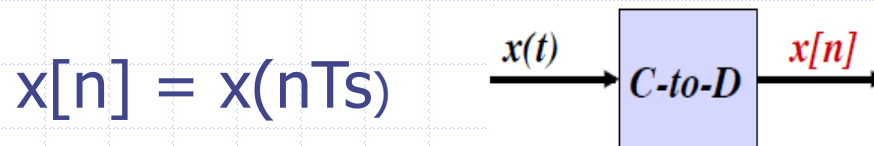


7.7 Application of The Fourier Transform

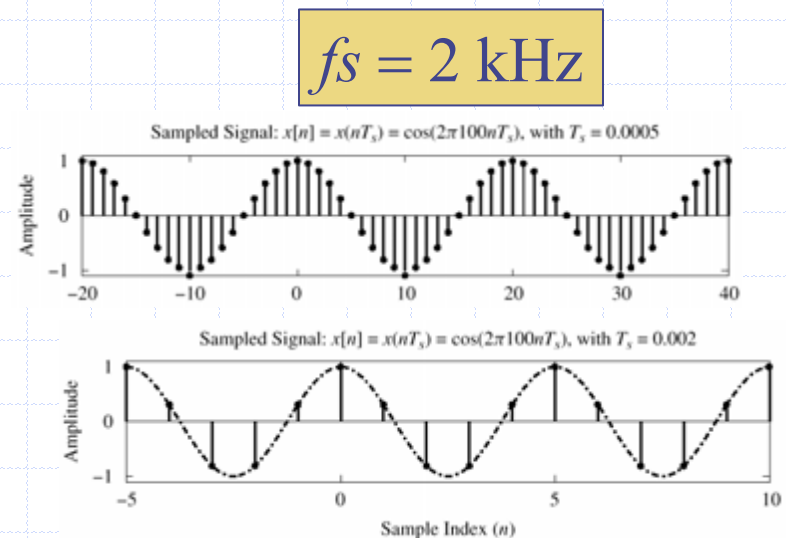
Sampling Process



- Use A-to-D converters to turn $x(t)$ into numbers $x[n]$
- Take a sample every sampling period T_s —uniform sampling



$$f = 100\text{Hz}$$



$$f_s = 500\text{Hz}$$

7.7 Application of The Fourier Transform

Sampling Theorem

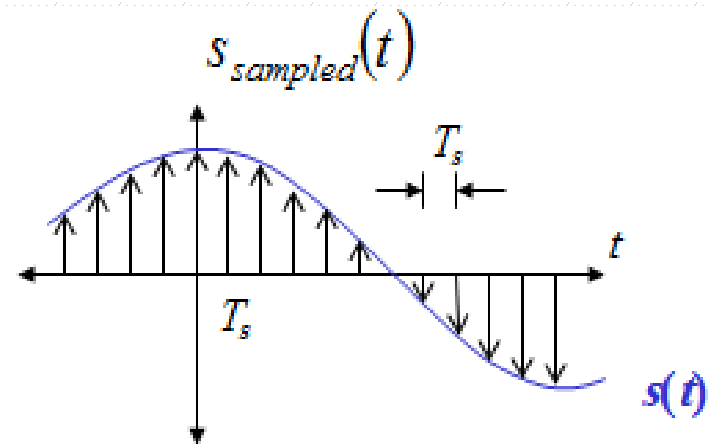
- Many signals originate as continuous-time signals, e.g. conventional music or voice
- By sampling a continuous-time signal at isolated, equally-spaced points in time, we obtain a sequence of numbers $S(k) = S(kT_s)$

$$n \in \{\dots, -2, -1, 0, 1, 2, \dots\}$$

T_s is the sampling period.

$$s_{\text{sampled}}(t) = s(t) \underbrace{\sum_{n=-\infty}^{\infty} \delta(t - nT_s)}_{\text{impulse train}}$$

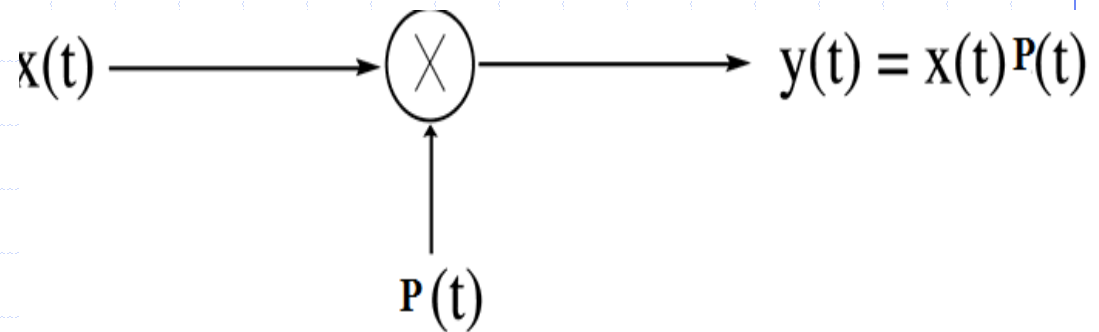
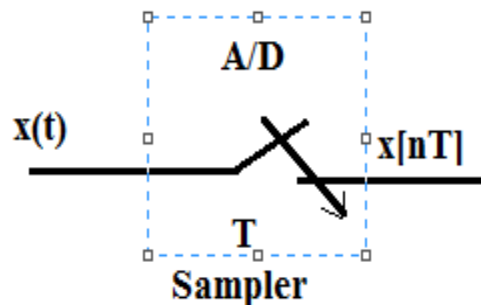
impulse train



Sampled analog waveform

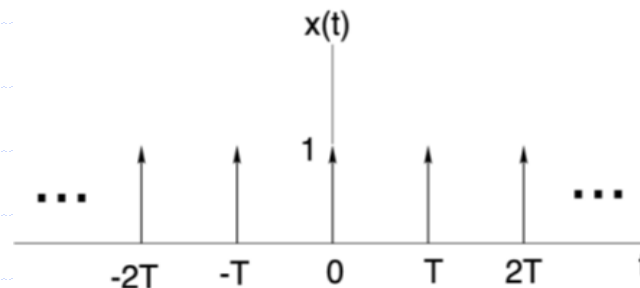
7.7 Application of The Fourier Transform

The Sampling Theorem A/D



Impulse modulation
mode

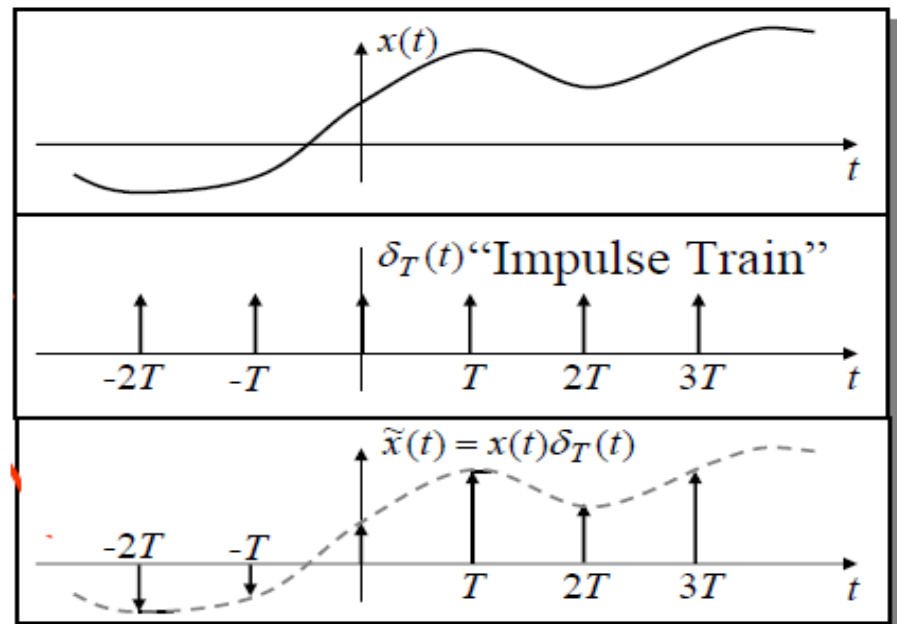
$$P(t) = \sum_{n=-\infty}^{n=\infty} \delta(t - nT)$$



7.7 Application of The Fourier Transform

$$P(t) = \sum_{n=-\infty}^{n=\infty} \delta(t - nT)$$

$$x_s(t) = x(t)p(t)$$



Because of the sampling property:

$$x_s(t) = x(t)P(t) = \sum_{n=-\infty}^{n=\infty} x(t)\delta(t - nT)$$

7.7 Application of The Fourier Transform

Shannon Sampling Theorem: A continuous-time signal $x(t)$ can be uniquely reconstructed from its samples $x_s(t)$ with two conditions:

- ❑ $x(t)$ must be band-limited with a maximum frequency ω_B
- ❑ Sampling frequency ω_s of $x_s(t)$ must be greater than $2\omega_B$, i.e. $\omega_s > 2\omega_B$.

The second condition is also known as *Nyquist Criterion*.

ω_s is referred as *Nyquist Frequency*, i.e. the smallest possible sampling frequency in order to recover the original analog signal from its samples.

So, in order to reconstruct an analog signal,:

- ◆ The first condition tells that $x(t)$ must not be changing fast.
- ◆ The second condition tells that we need to sample fast enough.

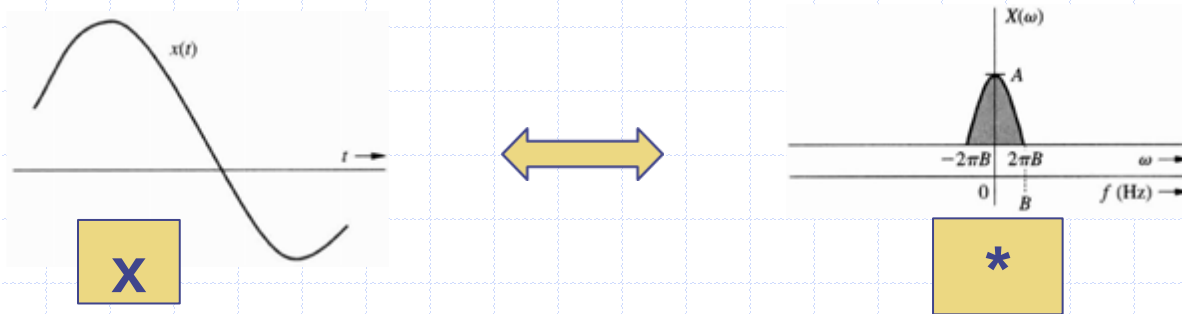
7.7 Application of The Fourier Transform

Generalized Sampling Theorem

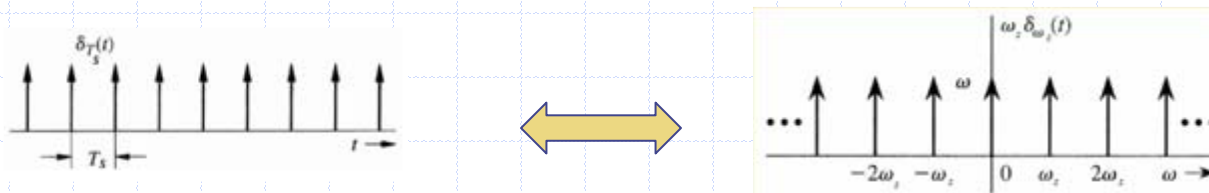
- Sampling rate must be greater than twice the **bandwidth**
- **Bandwidth** is defined as non-zero extent of spectrum of continuous-time signal in positive frequencies
- For **lowpass** signal with maximum frequency f_{max} , bandwidth is f_{max}
- For a **bandpass** signal with frequency content on the interval $[f_1, f_2]$, bandwidth is $f_2 - f_1$

7.7 Application of The Fourier Transform

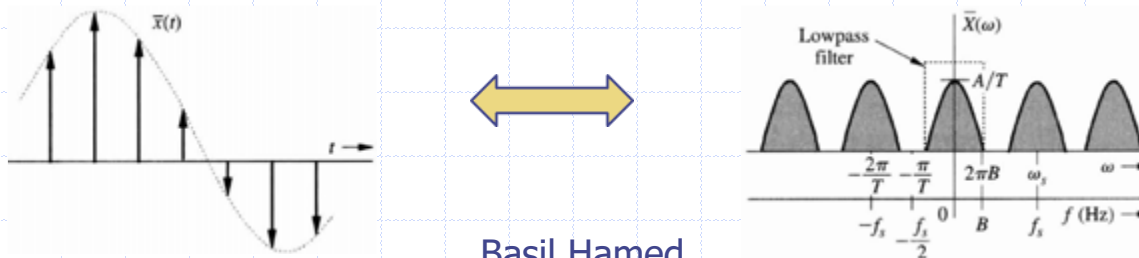
Consider a bandlimited signal $x(t)$ and its spectrum $X(\omega)$:



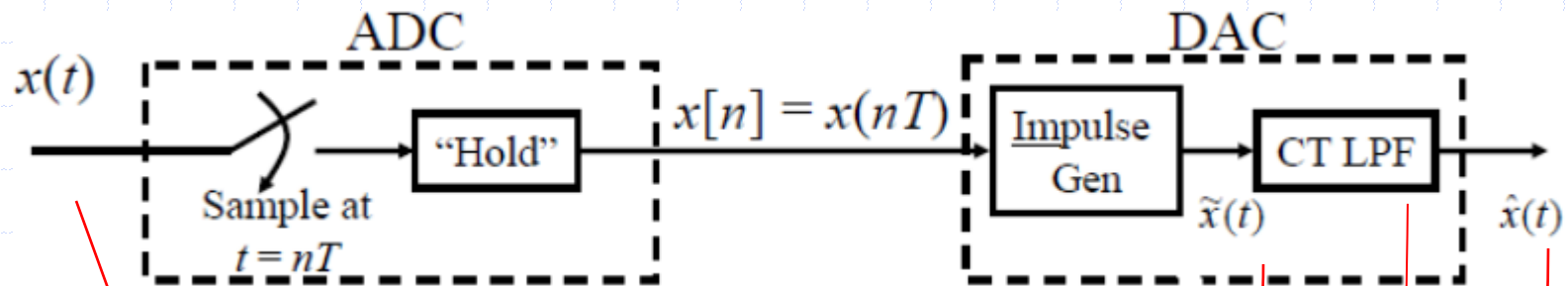
Ideal sampling = multiply $x(t)$ with impulse train



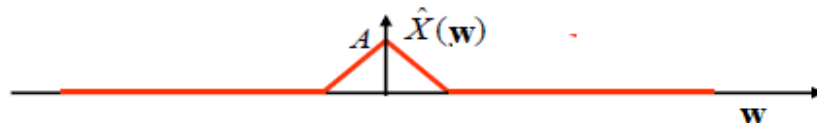
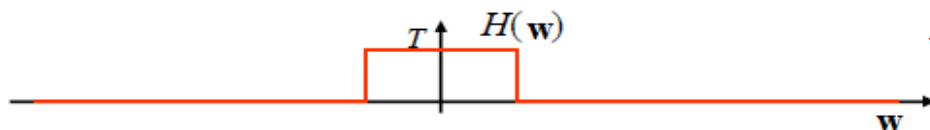
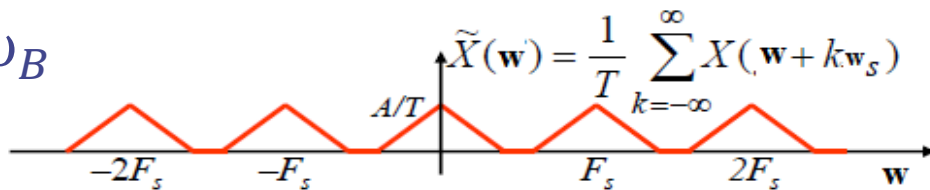
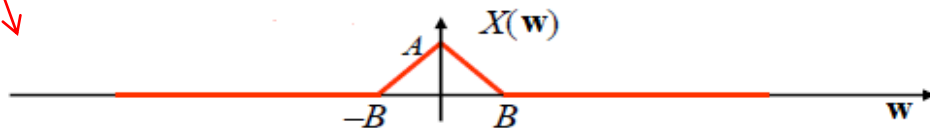
Therefore the sampled signal has a spectrum:



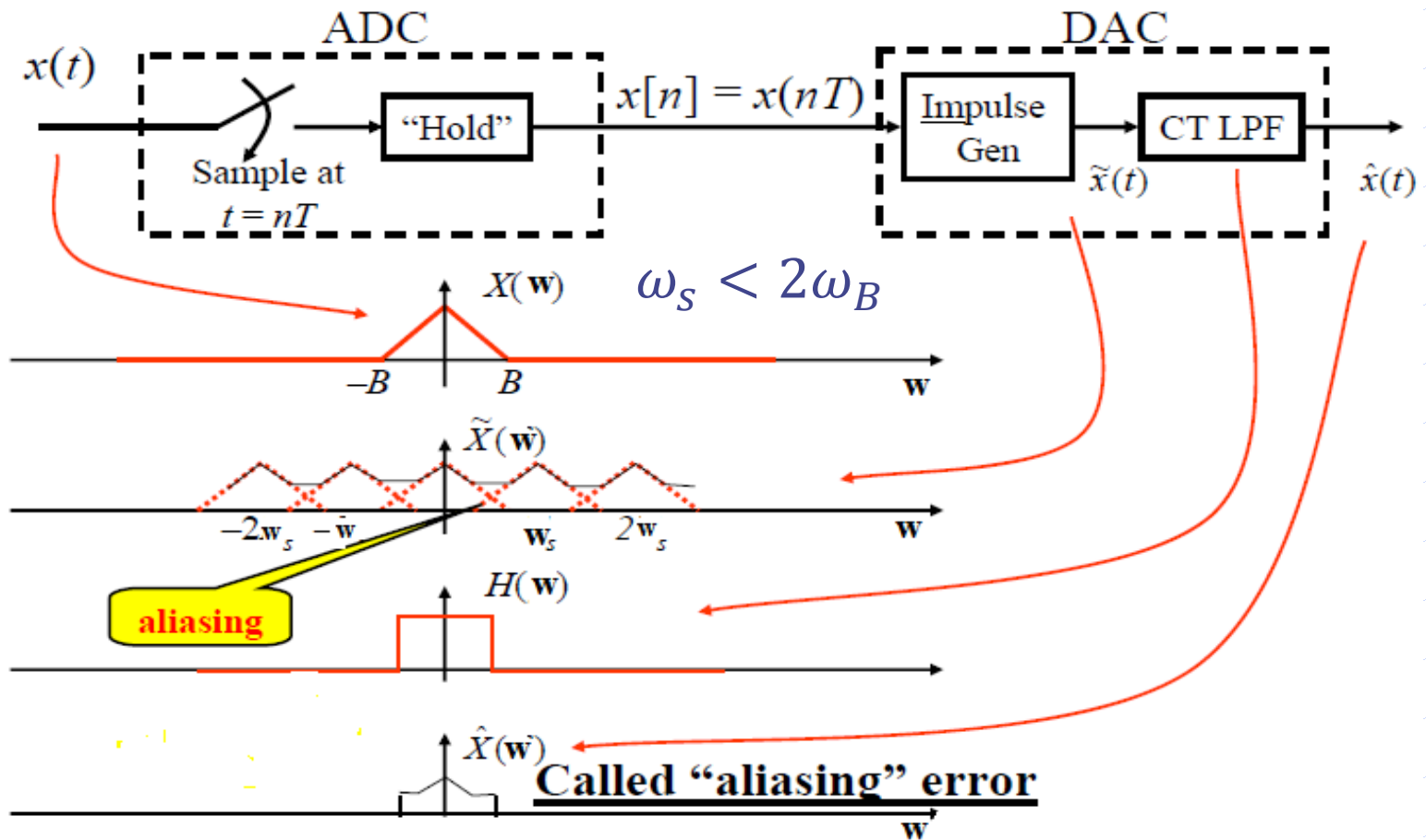
7.7 Application of The Fourier Transform



$$\omega_s \geq 2\omega_B$$



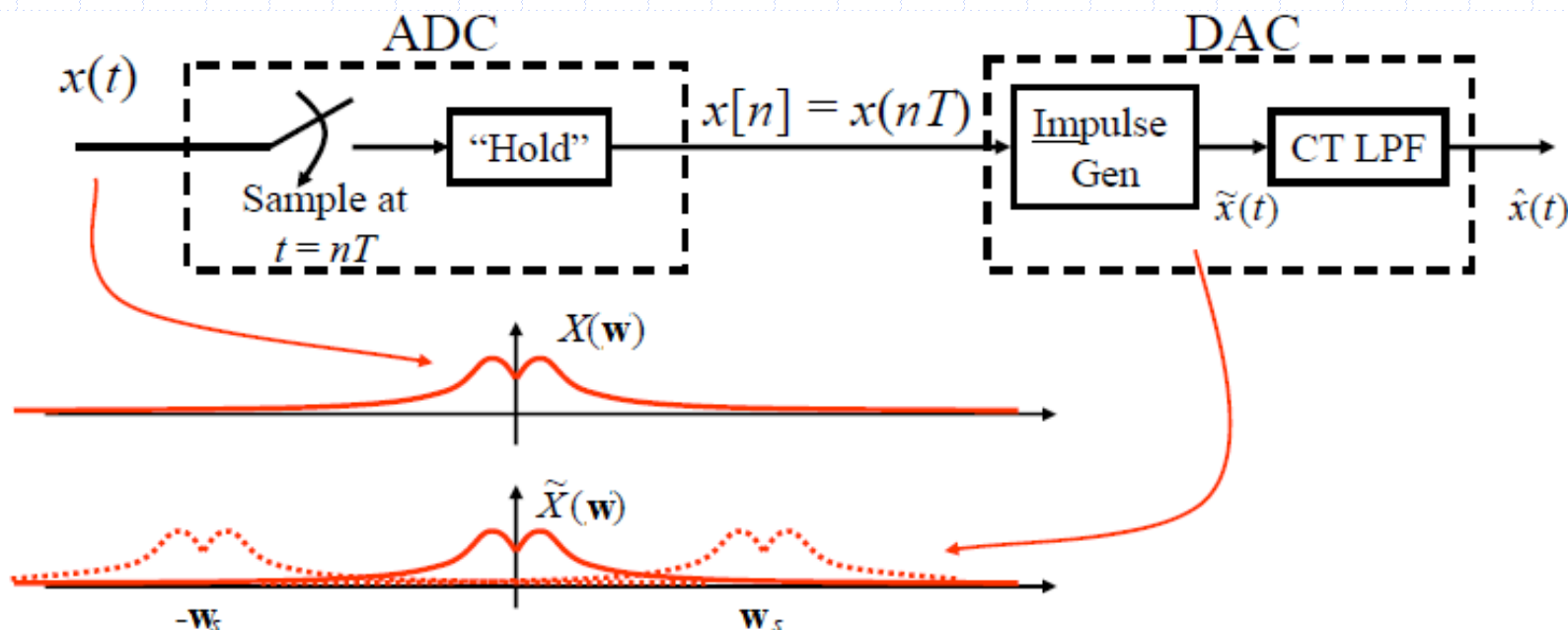
7.7 Application of The Fourier Transform



To enable error-free reconstruction, a signal bandlimited to ω_B Hz must be sampled faster than $2\omega_B$ samples/sec

7.7 Application of The Fourier Transform

“Aliasing” Analysis: What if the signal is NOT BANDLIMITED??

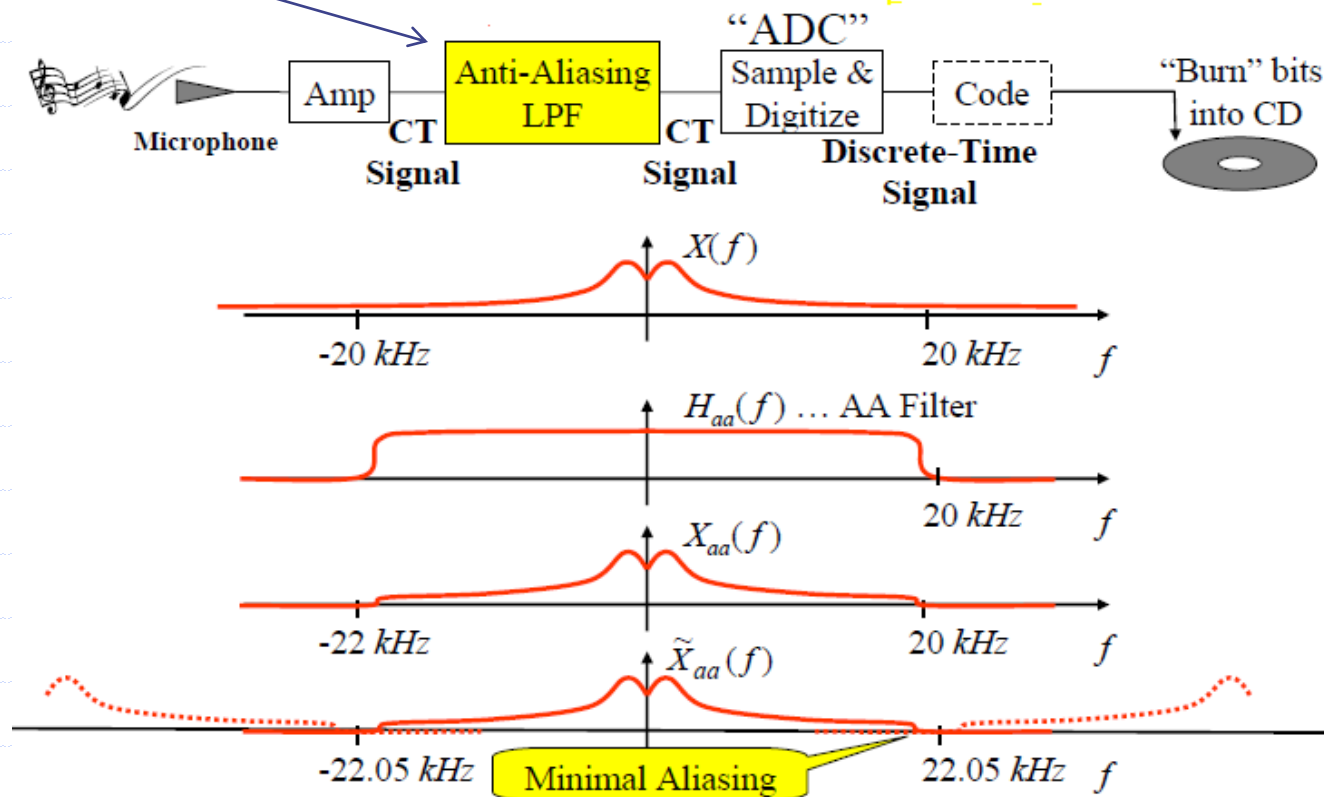


For Non-BL Signal Aliasing always happens regardless of ω_s value

7.7 Application of The Fourier Transform

Practical Sampling: Use of Anti-Aliasing Filter

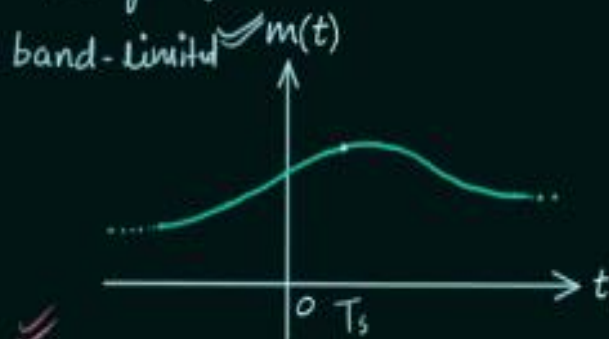
In practice it is important to avoid excessive aliasing. So we use a CT lowpass BEFORE the ADC!!!



✓ Sampling Theorem

Sampling: Reduction of a continuous time signal to a discrete time signal.

Message signal



$\omega_m = \text{max. freq. comp.}$



NESO ACADEMY

multiplier

SAMPLER

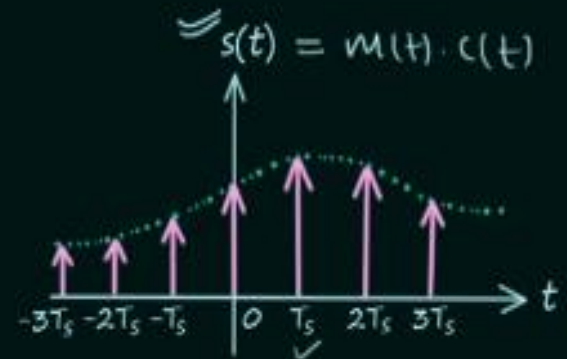


$$c(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$



$T_s = \text{sampling period / sampling int.}$

$$\omega_s = \frac{2\pi}{T_s} = \text{Sampling freq.}$$



$$s(t) \Rightarrow S(\omega)$$

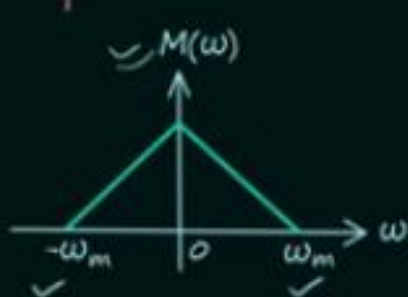
$$s(t) = m(t) \cdot c(t)$$

FT

$$S(\omega) = \frac{1}{2\pi} [M(\omega) * C(\omega)]$$

$$C(\omega) = \omega_s \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_s)$$

Sampling Theorem

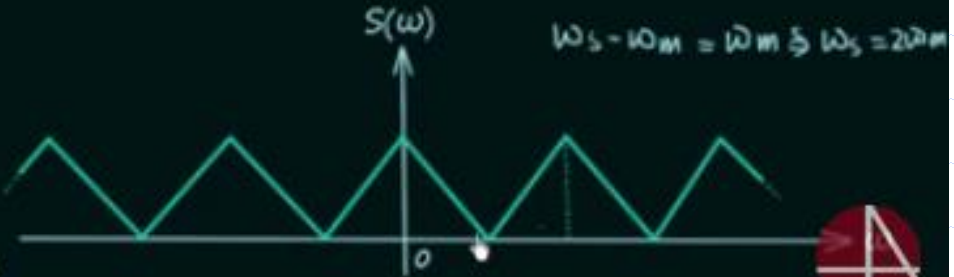
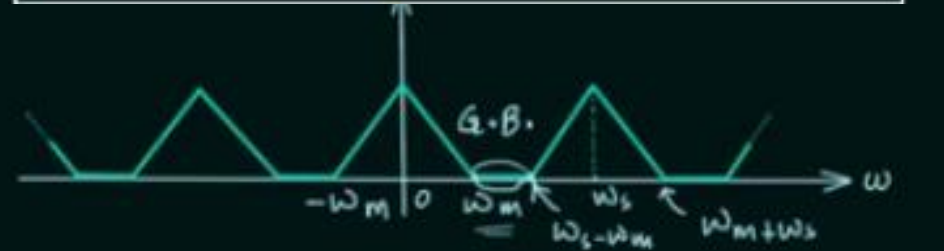
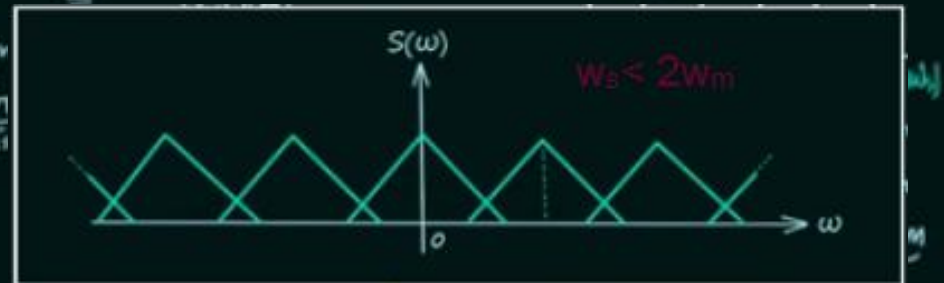


$$T_s = \text{sam}$$

$$\omega_s = \frac{2\pi}{T_s}$$

↓ FT

$$S(\omega) = \frac{1}{2\pi} [M(\omega) * (\omega_s)]$$



$$S(\omega) = \frac{1}{2\pi} [M(\omega) * \omega_s \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_s)]$$

$$S(\omega) = \frac{\omega_s}{2\pi} [M(\omega) * \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_s)]$$

$$S(\omega) = \frac{1}{T_s} \left[\sum_{n=-\infty}^{\infty} M(\omega) * \delta(\omega - n\omega_s) \right]$$

$$x(t) * \delta(t - t_1) = x(t - t_1)$$

$$S(\omega) = \frac{1}{T_s} \left[\sum_{n=-\infty}^{\infty} M(\omega - n\omega_s) \right]$$

$\nwarrow n=0 \quad \nearrow n=1$

$$S(\omega) = \frac{1}{T_s} [\dots + M(\omega + \omega_s) + \underbrace{M(\omega)}_{n=0} + \underbrace{M(\omega - \omega_s)}_{n=1} + \dots]$$

Summary of Fourier Transform Operations

Operation	$x(t)$	$X(\omega)$
Scalar multiplication	$kx(t)$	$kX(\omega)$
Addition	$x_1(t) + x_2(t)$	$X_1(\omega) + X_2(\omega)$
Conjugation	$x^*(t)$	$X^*(-\omega)$
Duality	$X(t)$	$2\pi x(-\omega)$
Scaling (a real)	$x(at)$	$\frac{1}{ a } X\left(\frac{\omega}{a}\right)$
Time shifting	$x(t - t_0)$	$X(\omega)e^{-j\omega t_0}$
Frequency shifting (ω_0 real)	$x(t)e^{j\omega_0 t}$	$X(\omega - \omega_0)$

Summary of Fourier Transform Operations

Operation	$x(t)$	$X(\omega)$
Time convolution	$x_1(t) * x_2(t)$	$X_1(\omega) X_2(\omega)$
Frequency convolution	$x_1(t) x_2(t)$	$\frac{1}{2\pi} X_1(\omega) * X_2(\omega)$
Time differentiation	$\frac{d^n x}{dt^n}$	$(j\omega)^n X(\omega)$
Time integration	$\int_{-\infty}^t x(u) du$	$\frac{X(\omega)}{j\omega} + \pi X(0) \delta(\omega)$

Final Exam 2008

A signal has Fourier transform ; $X(\omega) = \frac{\omega^2 + j4\omega + 2}{-\omega^2 + j4\omega + 3}$

Find the transform of $x(t) * \delta(t-1)$

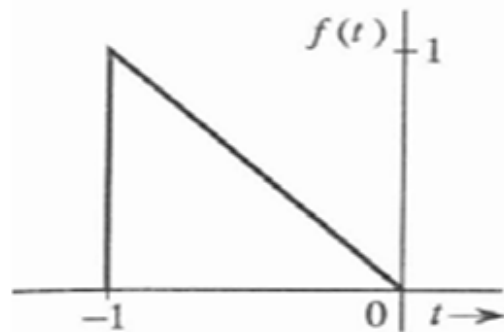
Solution

$$\mathcal{F}\{x(t) * \delta(t-1)\} = e^{-j\omega} X(\omega) = e^{-j\omega} \frac{\omega^2 + j4\omega + 2}{-\omega^2 + j4\omega + 3}$$

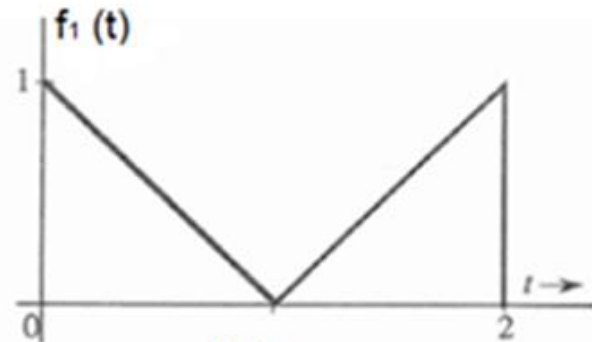
Final Exam 2017

The Fourier transform of the triangular pulse $f(t)$ shown in Fig (a) below is given to be

$$F(\omega) = \frac{1}{\omega^2} (e^{j\omega} - j\omega e^{j\omega} - 1)$$



(a)



(b)

Use this information and Fourier properties, find the Fourier transform of the signal shown in Fig (b)

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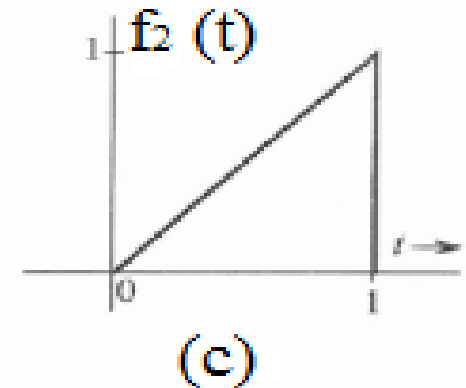


Fig (c) $f_2(t) = f(-t)$

$$\therefore F_2(\omega) = F(-\omega) = \frac{1}{\omega^2} [e^{-j\omega} + j\omega e^{-j\omega} - 1]$$

Fig (b) $f_1(t) = f(t-1) + f_2(t-1)$

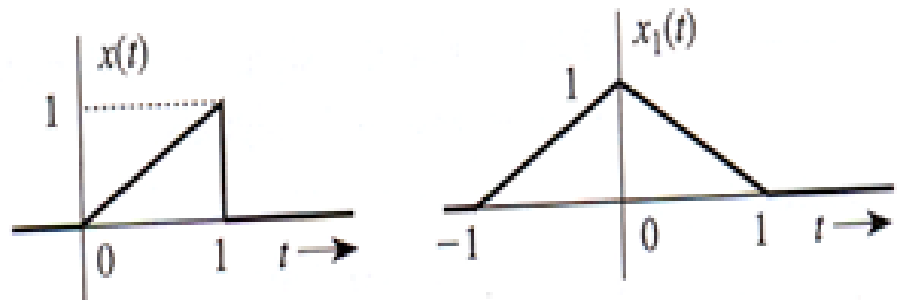
$$\begin{aligned}\therefore F_1(\omega) &= [F(\omega) e^{-j\omega} + F_2(\omega) e^{-j\omega}] \\ &= [F(\omega) + F(-\omega)] e^{-j\omega} \\ &= \frac{2}{\omega^2} (\cos \omega + \omega \sin \omega - 1)\end{aligned}$$

Final Exam 2016

The Fourier transform of the triangular pulse $x(t)$ in Fig. is expressed as

$$X(\omega) = \frac{1}{\omega^2} (e^{j\omega} - j\omega e^{j\omega} - 1)$$

Use this information, and Fourier properties, to find the Fourier transform of the signal $x_1(t)$ shown in Fig. below.



$$x_1(t) = x(t+1) + x(1-t)$$

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$$x[k+1] \longleftrightarrow X(\omega) e^{j\omega}$$

$$x[1-k] \longleftrightarrow X(-\omega) e^{-j\omega}$$

$$x_1[k] = x[k+1] + x[1-k]$$

$$X_1(\omega) = X(\omega) e^{j\omega} + X(-\omega) e^{-j\omega}$$

$$X_1(\omega) = \frac{1}{\omega^2} (e^{j\omega} - j\omega e^{j\omega} - 1) e^{j\omega} + \frac{1}{\omega^2} (e^{-j\omega} + j\omega e^{-j\omega} - 1) e^{-j\omega}$$

$$x_1(\omega) = \frac{1}{\omega^2} (2 \cos \omega - 2)$$