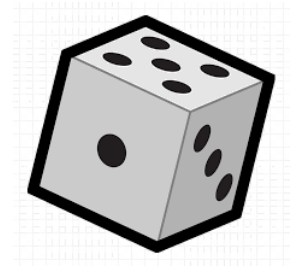


1.2 Applying Set Theory to Probability

Random experiment

Random experiment: A phenomenon whose outcome cannot be predicted with certainty, such as

- Roll a die
- Roll a die three times
- Flip a coin
- Transmit one of a collection of waveforms over a channel. What waveform does arrive at the receiver?



Outcome:

An outcome is the result of a random experiment.

- Roll a die $\longrightarrow$ 3
- Roll a die 3 times $\longrightarrow$ (2, 3, 6)

1.2 Applying Set Theory to Probability

Random experiment

Events: An event is a set of outcomes of an experiment (one (or more) outcomes.)

1- Flip a coin: observe head or tail

- Two possible outcomes (heads (H) or tails (T))
- Sample space: $S = \{H, T\}$



2- Flip a coin 3 times: observe the sequence of heads and tails head or tail

- Sample space: $S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$

possible outcomes

- Possible Event: to have 2 or more heads $E = \{HHH, HHT, HTH, THH\}$

3- Flip a coin 3 times: observe the number of heads

- **Sample space:** $S = \{0, 1, 2, 3\}$

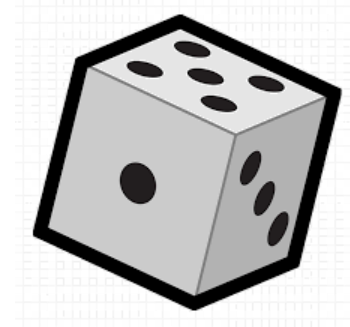
1.2 Applying Set Theory to Probability

Random experiment

4- Roll a die:  random experiment

$$S = \{1, 2, 3, 4, 5, 6\}$$

- Possible Event: the roll is even $E = \{2, 4, 6\}$



5- Roll a die three times

$$S = \{(1, 1, 1), (1, 1, 2), \dots, (1, 1, 6), (2, 1, 1), \dots, (6, 6, 6)\}$$

$\Rightarrow 6^3$ elements

 an outcome

- Possible Event: All are equal

$$E = \{(1, 1, 1), (2, 2, 2), (3, 3, 3), (4, 4, 4), (5, 5, 5), (6, 6, 6)\}$$

1.3 Axioms of Probability

Definition. A probability measure $P(\cdot)$ is a function that maps events in the sample space S to real numbers. Such that:

Axiom 1: For any event A , $P(A) \geq 0$.

Axiom 2: Probability of the sample space S is $P(S) = 1$.

Axiom 3: For any countable collection $A_1, A_2, A_3, \dots$ of disjoint events

$$P(A_1 \cup A_2 \cup A_3 \dots) = P(A_1) + P(A_2) + P(A_3) + \dots$$

1.3 Axioms of Probability

Theorems

- For mutually exclusive events A_1 and A_2

$$P(A_1 \cup A_2) = P(A_1) + P(A_2)$$

- If $A = (A_1 \cup A_2 \cup \dots \cup A_m)$ and $A_i \cap A_j = \emptyset$ for $i \neq j$

$$P(A) = \sum_{i=1}^m P[A_i]$$

- The probability of an event $B = \{s_1, s_2, \dots, s_m\}$ is the sum of the probabilities of the outcomes contained in the event:

$$P(B) = \sum_{i=1}^m P[\{s_i\}]$$

1.3 Axioms of Probability

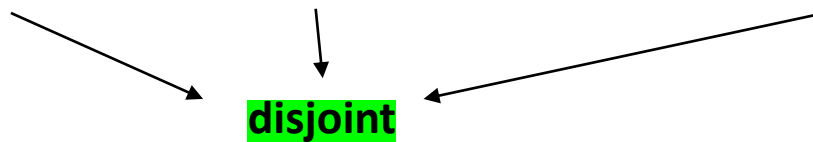
Theorems

- For an experiment with sample space $S = \{s_1, s_2, \dots, s_n\}$ in which each outcome s_i is equally likely,

$$P(s_i) = \frac{1}{n} \quad 1 \leq i \leq n$$

Roll a fair die (fair: outcomes are equally likely).

$$P(\{1\}) = P(\{2\}) = \dots = P(\{6\})$$



$$P(\{1\} \cup \{2\} \cup \{3\} \cup \dots \cup \{6\}) = P(\{1\}) + P(\{2\}) + \dots + P(\{6\}) = 6P(\{1\})$$

$$1 = P(S) = P(\{1, 2, \dots, 6\}) \Rightarrow P(1) = P(2) = \dots = P(6) = \frac{1}{6}$$

1.3 Axioms of Probability

Examples

- An urn contains 10 identical balls numbered 0,1,2,...9. A random experiment involves selecting a ball from the urn and noting the number of the ball. Find the probability of the following events:
 - A : number of ball selected is odd
 - B : number of ball selected is a multiple of 3
 - C : number of ball selected is less than 5

$$S = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

$$A = \{1, 3, 5, 7, 9\} \Rightarrow P[A] = P[\{1\}] + P[\{3\}] + P[\{5\}] + P[\{7\}] + P[\{9\}] = \frac{5}{10}.$$

$$B = \{3, 6, 9\} \Rightarrow P[B] = P[\{3\}] + P[\{6\}] + P[\{9\}] = \frac{3}{10}.$$

$$C = \{0, 1, 2, 3, 4\} \Rightarrow P[C] = P[\{0\}] + P[\{1\}] + P[\{2\}] + P[\{3\}] + P[\{4\}] = \frac{5}{10}.$$

1.3 Axioms of Probability

Using the axioms:

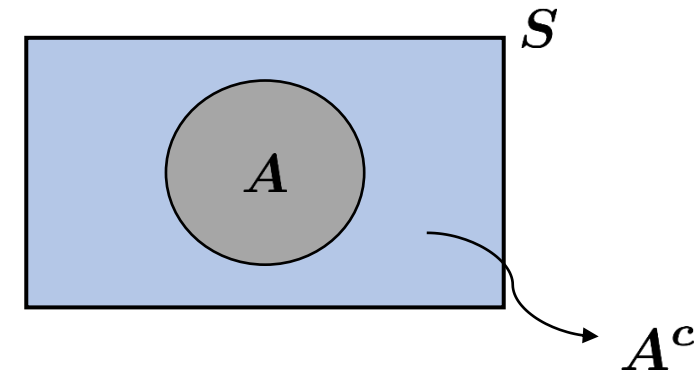
1) what is $P(A^c)$?

$$A \cup A^c = S$$

$$\Rightarrow P(A \cup A^c) = P(S) = 1$$

$\swarrow \quad \searrow$
disjoint

$$\Rightarrow P(A) + P(A^c) = 1 \Rightarrow P(A^c) = 1 - P(A)$$



1.3 Axioms of Probability

Using the axioms:

2) $P(\emptyset) = 0,$

$\emptyset$: empty

$$P(\emptyset) = P(S^c) = 1 - P(S) = 1 - 1 = 0.$$

3) $P(A) \leq 1,$

$$P(A) = 1 - \underbrace{P(A^c)}_{\geq 0} \Rightarrow P(A) \leq 1.$$

(the first axiom)

1.3 Axioms of Probability

Using the axioms:

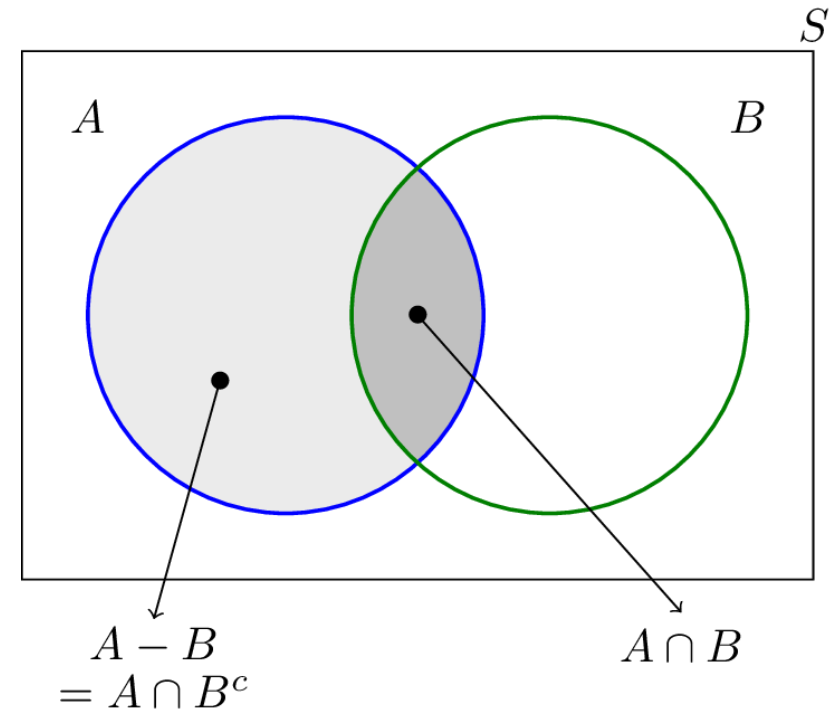
$$4) \quad P(A - B) = P(A) - P(A \cap B),$$

$$A = (A \cap B) \cup (A - B)$$

$(A \cap B)$ and $(A - B)$ are disjoint.

$$\begin{aligned} P(A) &= P((A \cap B) \cup (A - B)) \\ &= P(A \cap B) + P(A - B). \end{aligned}$$

$$\rightarrow P(A - B) = P(A) - P(A \cap B).$$



1.3 Axioms of Probability

Using the axioms:

$$5) P(A \cup B) = P(A) + P(B) - P(A \cap B),$$

$\cup$: or , $\cap$: and

Use Venn diagram.

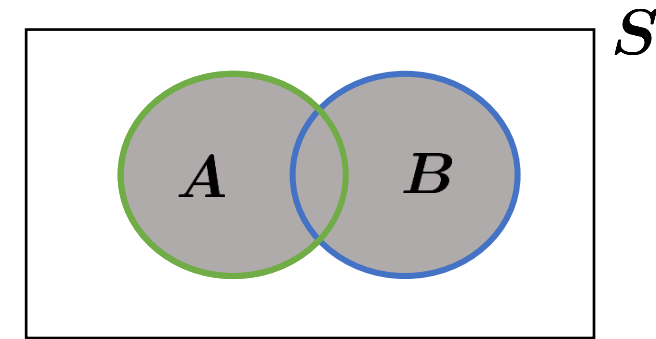
$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - \\ - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

$$6) A \subset B \Rightarrow P(A) \leq P(B),$$

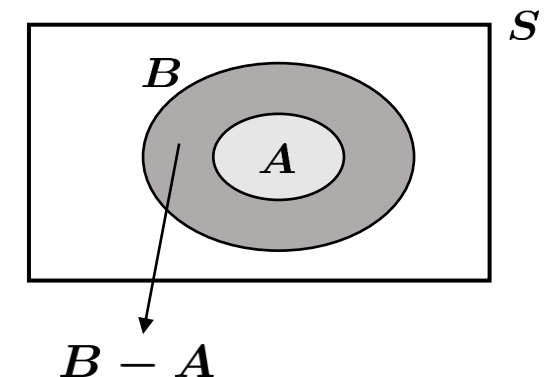
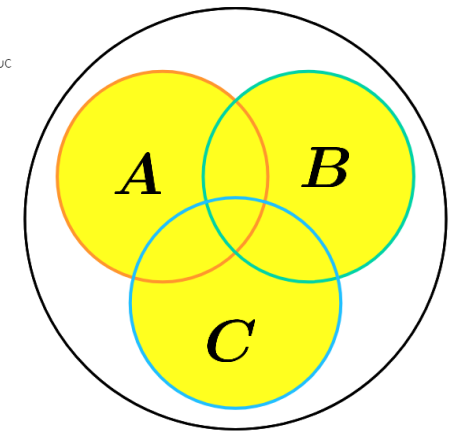
$$P(B) = P(A) + P(B - A)$$

$P(B - A) \geq 0$, B and $(A - B)$ are disjoint.

$$\Rightarrow P(B) \geq P(A)$$



Legend:
A (orange)
B (green)
C (blue)
A ∪ B ∪ C (yellow)

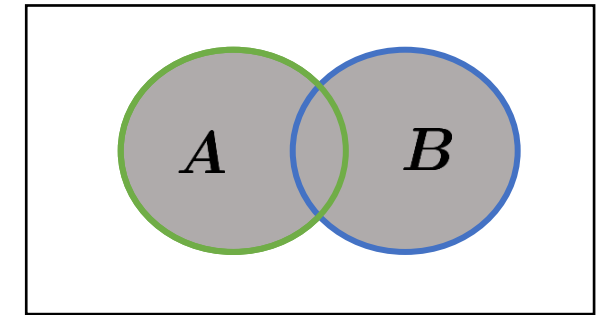


1.3 Axioms of Probability

Examples

- An urn contains 10 identical balls numbered 0,1,2,...9. A random experiment involves selecting a ball from the urn and noting the number of the ball. Find the probability of the following events:

- A : number of ball selected is odd
- B : number of ball selected is a multiple of 3



$$S = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

$$A = \{1, 3, 5, 7, 9\} \Rightarrow P[A] = P[\{1\}] + P[\{3\}] + P[\{5\}] + P[\{7\}] + P[\{9\}] = \frac{5}{10}.$$

$$B = \{3, 6, 9\} \Rightarrow P[B] = P[\{3\}] + P[\{6\}] + P[\{9\}] = \frac{3}{10}.$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(\{A \cap B\}) = P(\{3, 9\}) = \frac{2}{10}$$

$$P(A \cup B) = \frac{5}{10} + \frac{3}{10} - \frac{2}{10} = \frac{6}{10}$$

$$\{(A \cup B) = \{1, 3, 5, 6, 7, 9\}\}$$

1.3 Axioms of Probability

Examples

Let A, B, C be three events in the sample space S . Suppose we know:

1) $A \cup B \cup C = S$

2) $P(A) = \frac{1}{2}$

3) $P(B) = \frac{2}{3}$

4) $P(A \cup B) = \frac{5}{6}$

Answer the following questions:

A) $P(A \cap B)$?

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\frac{5}{6} = \frac{1}{2} + \frac{2}{3} - P(A \cap B)$$

$$P(A \cap B) = \frac{1}{3}$$

B) Do A, B, C form a partition of S ?

not disjoint

No, since $A \cap B \neq \emptyset$

1.3 Axioms of Probability

Examples

Let A, B, C be three events in the sample space S . Suppose we know:

1) $A \cup B \cup C = S$

2) $P(A) = \frac{1}{2}$

3) $P(B) = \frac{2}{3}$

4) $P(A \cup B) = \frac{5}{6}$

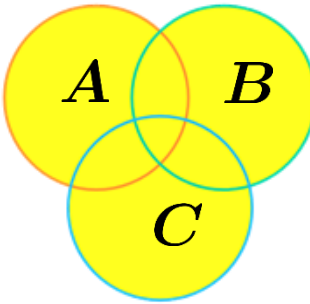
Answer the following questions:

c) $P(C - (A \cup B))$?

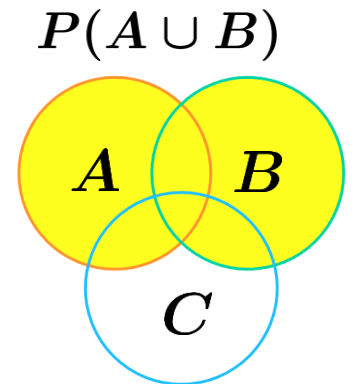
$$\begin{aligned} C - (A \cup B) &= \left(C \cup (A \cup B) \right) - (A \cup B) \\ &= S - (A \cup B) \quad (\text{since } A \cup B \cup C = S) \\ &= (A \cup B)^c. \end{aligned}$$

$$\begin{aligned} P(C - (A \cup B)) &= P((A \cup B)^c) \\ &= 1 - P(A \cup B) \\ &= \frac{1}{6}. \end{aligned}$$

A
B
C
A ∪ B ∪ C



A
B
C
A ∪ B



1.3 Axioms of Probability

Examples

Let A, B, C be three events in the sample space S . Suppose we know:

1) $A \cup B \cup C = S$

2) $P(A) = \frac{1}{2}$

3) $P(B) = \frac{2}{3}$

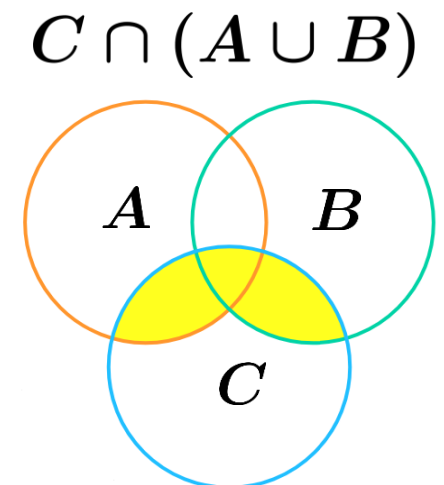
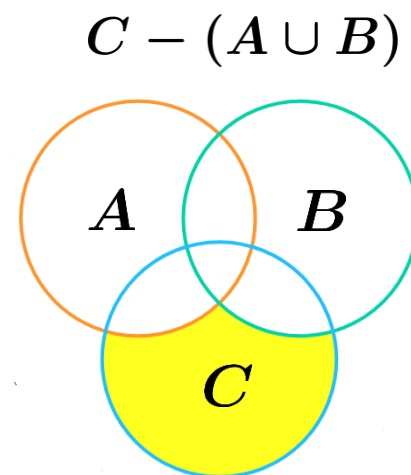
4) $P(A \cup B) = \frac{5}{6}$

Answer the following questions:

D) If $P(C \cap (A \cup B)) = \frac{5}{12}$, Find $P(C)$

$$P(C) = P(C \cap (A \cup B)) + P(C - (A \cup B))$$

$$= \frac{5}{12} + \frac{1}{6} = \frac{7}{12}.$$

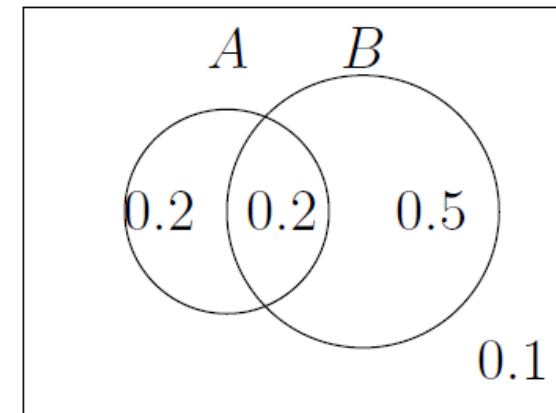


1.3 Axioms of Probability

Examples

Let A and B be two events such that

$$P(A) = 0.4, P(B) = 0.7, P(A \cup B) = 0.9$$



- a. Find $P(A \cap B)$.
- b. Find $P(A^c \cap B)$.
- c. Find $P(A - B)$.
- d. Find $P(A^c - B)$.
- e. Find $P(A^c \cup B)$.
- f. Find $P(A \cap (B \cup A^c))$.

a)
$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$0.9 = 0.7 + 0.4 - P(A \cap B)$$

$$P(A \cap B) = 0.2$$

b)
$$\begin{aligned} P(A^c \cap B) &= P(B - A) \\ &= P(B) - P(B \cap A) \\ &= 0.7 - 0.2 \\ &= 0.5 \end{aligned}$$

1.3 Axioms of Probability

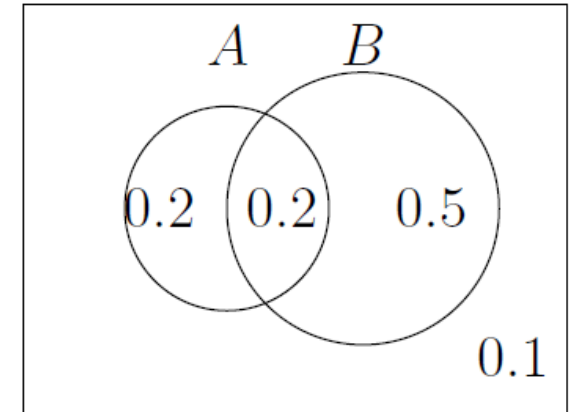
Examples

Let A and B be two events such that

$$P(A) = 0.4, P(B) = 0.7, P(A \cup B) = 0.9$$

- a. Find $P(A \cap B)$.
- b. Find $P(A^c \cap B)$.
- c. Find $P(A - B)$.
- d. Find $P(A^c - B)$.
- e. Find $P(A^c \cup B)$.
- f. Find $P(A \cap (B \cup A^c))$.

- c) $P(A - B) = 0.2$
- d) $P(A^c - B) = P(A^c \cap B^c)$
 $= P((A \cup B)^c)$
 $= 1 - P(A \cup B)$
 $= 1 - 0.9$
 $= 0.1$

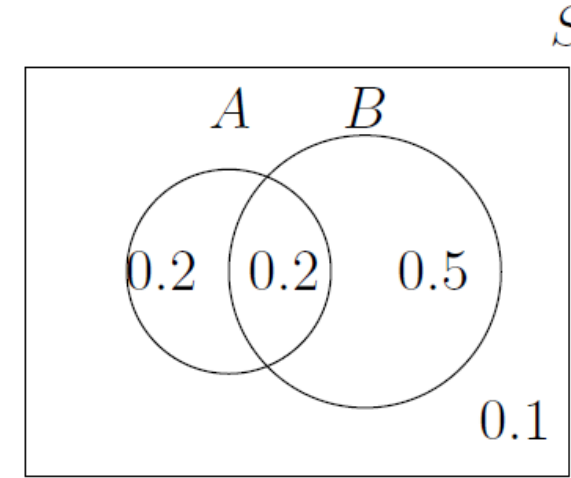


1.3 Axioms of Probability

Examples

Let A and B be two events such that

$$P(A) = 0.4, P(B) = 0.7, P(A \cup B) = 0.9$$



- a. Find $P(A \cap B)$.
- b. Find $P(A^c \cap B)$.
- c. Find $P(A - B)$.
- d. Find $P(A^c - B)$.
- e. Find $P(A^c \cup B)$.
- f. Find $P(A \cap (B \cup A^c))$.

$$\begin{aligned} \text{e) } P(A^c \cup B) &= P(A^c) + P(B) - P(A^c \cap B) \\ &= 1 - P(A) + P(B) - P(B - A) \\ &= 1 - 0.4 + 0.7 - 0.5 \\ &= 0.8 \end{aligned}$$

f) Who send first by Email will get 1 mark Bonus.