

1.4 Conditional Probability

Introduction

Suppose that in a certain city, 0.3 percent of the days are **rainy**.

$$P(\text{Rain}) = 0.3$$

The probability that it **rains** given that it is **cloudy** might be:

$$P(\text{Rain}|\text{Cloudy}) = 0.9$$

Conditional probability: The probability A given B

$$P(A|B)$$

1.4 Conditional Probability

Introduction

Example. Roll a fair die, what is the probability that the outcome is an even number given it was less than or equal to 3, i.e.,

$$A = \{2, 4, 6\}$$

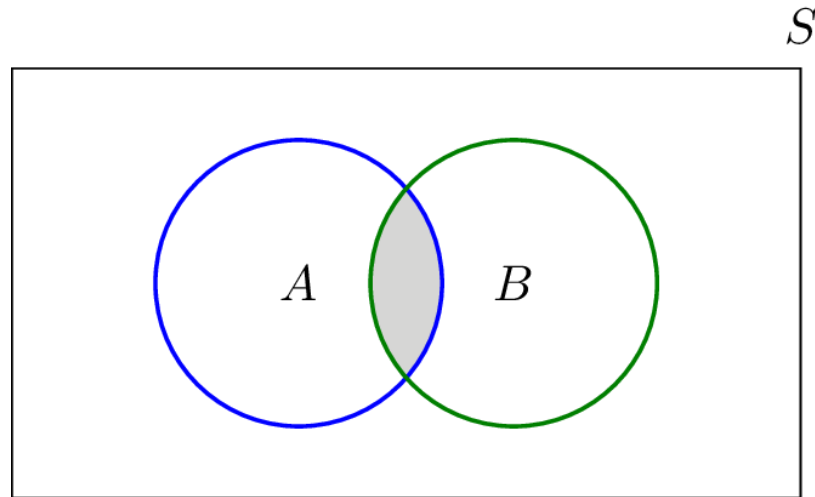
$$B = \{1, 2, 3\} \longrightarrow \text{has occurred}$$

1.4 Conditional Probability

Definition

The **Conditional Probability** $P(A|B)$, the probability that A occurred given that B has occurred is given by

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \text{ when } P(B) > 0.$$



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$$B = \{1, 2, 3\} \longrightarrow \text{has occurred}$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1}{3}$$

1.4 Conditional Probability

Definition

Conditional probability satisfies the probability axioms :

- a) For any event A , $P(A|B) \geq 0$.
- b) Conditional probability of B given B is $P(B|B) = 1$.
- c) If $A_1, A_2, A_3, \dots$ are disjoint events, then

$$P(A_1 \cup A_2 \cup A_3 \dots | B) = P(A_1|B) + P(A_2|B) + P(A_3|B) + \dots .$$

1.4 Conditional Probability

Example

Roll two dice X_1, X_2 .

A : 4 dots are shown at least on one die $X_1 = 4$ or $X_2 = 4$,

B : $X_1 + X_2 = 7$,

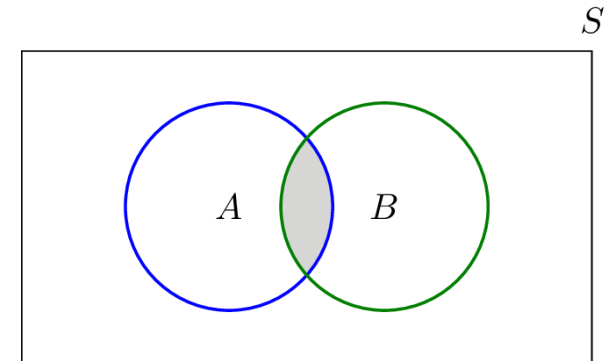
Find $P(A|B) = \frac{P(A \cap B)}{P(B)}$.

$$A = \{(4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6), (1, 4), (2, 4), (3, 4), (5, 4), (6, 4)\}$$

$$B = \{(6, 1), (5, 2), (4, 3), (3, 4), (2, 5), (1, 6)\}$$

$$A \cap B = \{(4, 3), (3, 4)\}$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{2}{36}}{\frac{6}{36}} = \frac{1}{3}.$$



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Definition

Special cases:

1) A and B are disjoint:

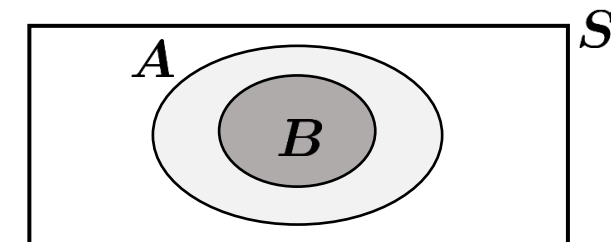
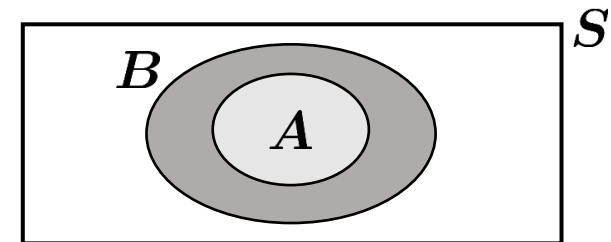
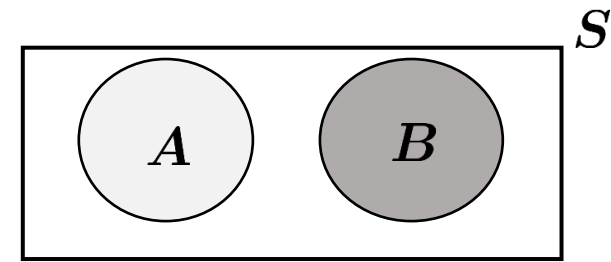
$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0}{P(B)} = 0.$$

2) $A \subset B$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(B)}.$$

3) $B \subset A$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1.$$

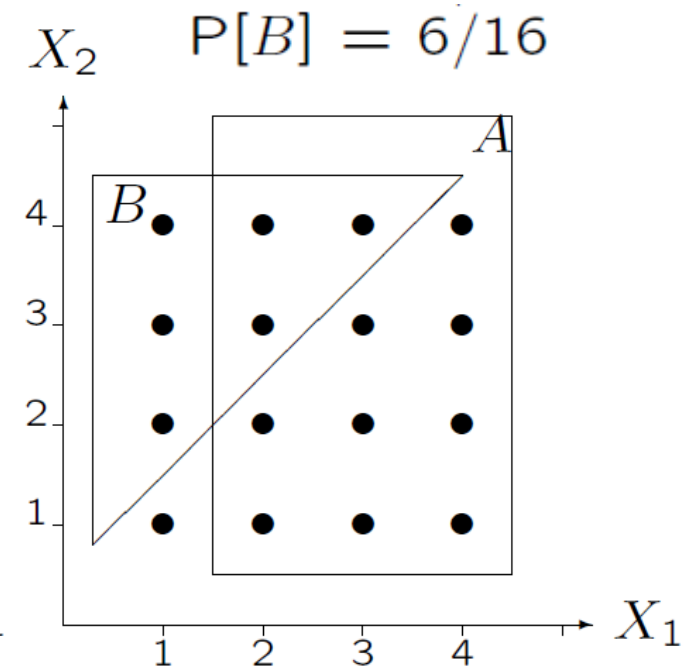
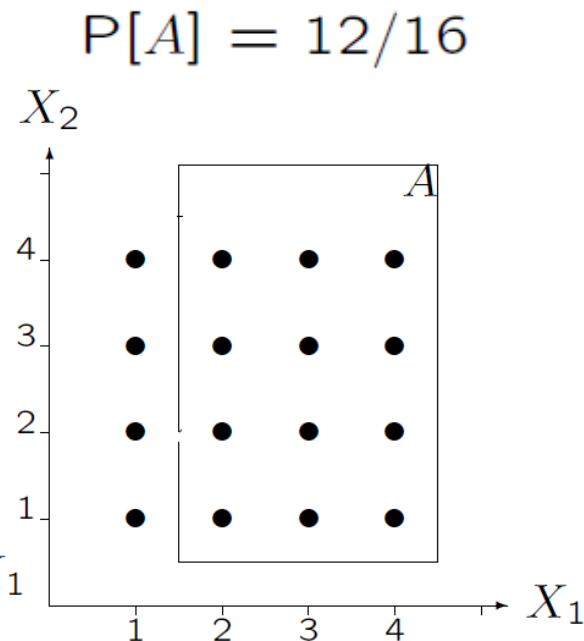
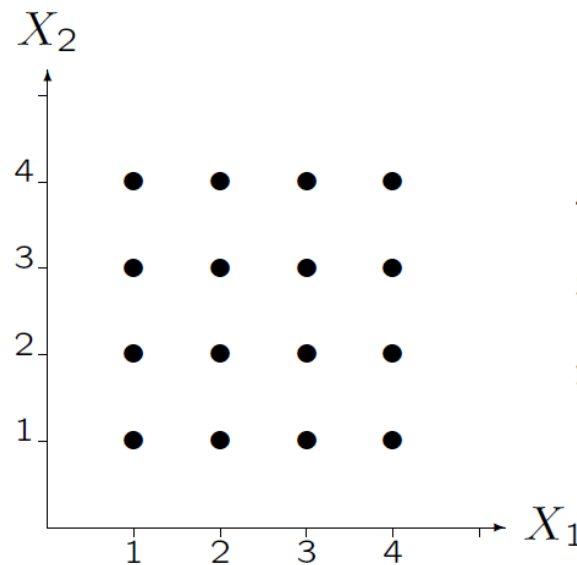


1.4 Conditional Probability

Example

Roll two fair four-sided dice. Let X_1 and X_2 denote the number of dots that appear on die 1 and die 2, respectively. Let A be the event $X_1 \geq 2$. What is $P[A]$? Let B denote the event $X_2 > X_1$. What is $P[B]$? What is $P[A|B]$?

$$P[A|B] = \frac{P(A \cap B)}{P[B]} = \frac{1}{2}.$$



1.4 Conditional Probability

Example

Mobile telephones perform *handoffs* as they move from cell to cell. During a call, a telephone either performs zero handoffs (H_0), one handoff (H_1), or more than one handoff (H_2). In addition, each call is either long (L), if it lasts more than three minutes, or brief (B). The following table describes the probabilities of the possible types of calls.

	H_0	H_1	H_2
L	0.1	0.1	0.2
B	0.4	0.1	0.1

What is the probability $P[H_0]$ that a phone makes no handoffs? What is the probability a call is brief? What is the probability a call is long or there are at least two handoffs?

$$P[H_0] = P[LH_0] + P[BH_0] = 0.1 + 0.4 = 0.5$$

$$P[B] = P[BH_0] + P[BH_1] + P[BH_2] = 0.4 + 0.1 + 0.1 = 0.6$$

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$$\begin{aligned} P[L \cup H_2] &= P[LH_0] + P[LH_1] + P[LH_2] + P[BH_2] \\ &= 0.1 + 0.1 + 0.2 + 0.1 = 0.5. \end{aligned}$$

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	H_0	H_1	H_2
L	0.1	0.1	0.2
B	0.4	0.1	0.1

What is the probability that a brief call will have no handoffs?

$$P[H_0|B] = \frac{P(H_0 \cap B)}{P[B]} = \frac{0.4}{0.6} = \frac{2}{3}.$$

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	H_0	H_1	H_2
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What is the probability that a call with one handoff will be long?

$$P[H_1] = P[H_1B] + P[H_1L] = 0.2 \quad P[L|H_1]$$

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	H_0	H_1	H_2
L	0.1	0.1	0.2
B	0.4	0.1	0.1

What is the probability that a long call will have one or more handoffs?

$$P[L] = 1 - P[B] = 0.4 \qquad P[H_1 \cup H_2|L] :$$