

1.5 Law of Total Probability

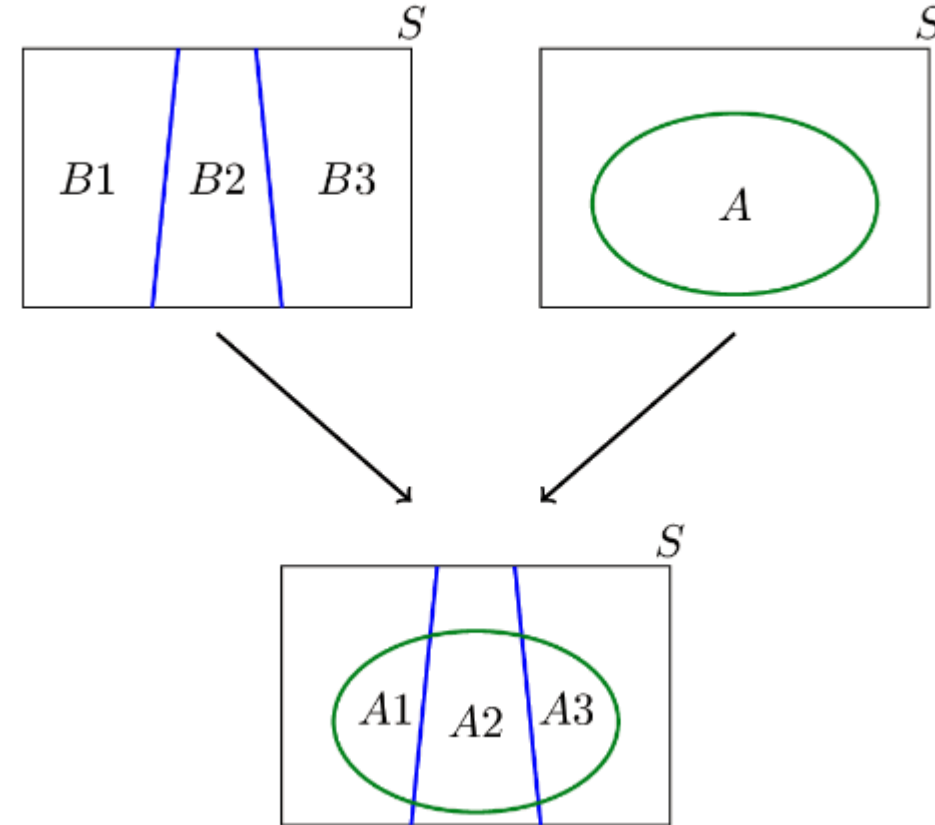
Introduction

$$A_i = A \cap B_i$$

$$P(A) = P(A_1) + P(A_2) + P(A_3) + \dots$$

$$P(A) = \sum_{i=1}^m P(A_i) = \sum_{i=1}^m P(A \cap B_i).$$

$$P(A|B_i) = \frac{P(A \cap B_i)}{P(B_i)} \Rightarrow P(A \cap B_i) = P(A|B_i)P(B_i).$$



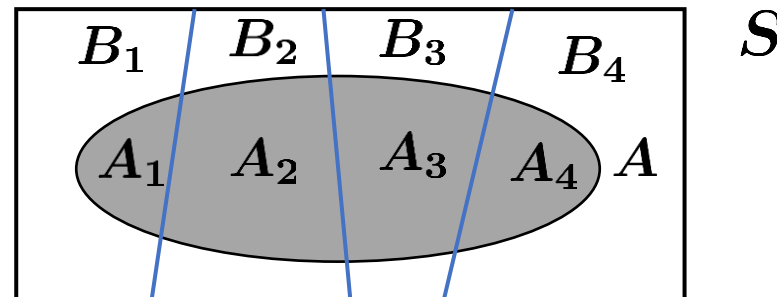
1.5 Law of Total Probability

Definition

Let $B_1, B_2, B_3, \dots$ be a **partition** of the sample space S with $P(B_i) > 0$.

For any event A we have

$$P(A) = \sum_{i=1}^m P(A \cap B_i) = \sum_{i=1}^m P(A|B_i)P(B_i).$$

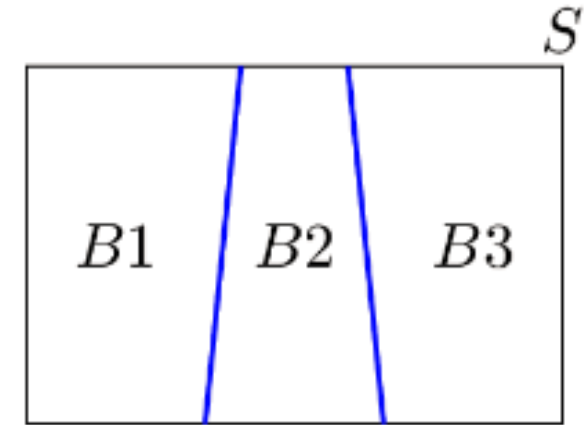


1.5 Law of Total Probability

Example 1.5.1

Three bags that each contain 100 marbles:

- Bag 1 has 75 red and 25 blue marbles;
- Bag 2 has 60 red and 40 blue marbles;
- Bag 3 has 45 red and 55 blue marbles.



One choose one of the bags at random and then pick a marble from the chosen bag, also at random. What is the probability that the chosen marble is red?

R : the event of selecting a red marble

B_i : the event that Bag i is selected

$$P(A) = \sum_{i=1}^m P(A \cap B_i) = \sum_{i=1}^m P(A|B_i)P(B_i).$$

$$P(R|B_1) = 0.75$$

$$P(R|B_2) = 0.60$$

$$P(R|B_3) = 0.45$$

$$\begin{aligned} P(R) &= P(R|B_1)P(B_1) + P(R|B_2)P(B_2) + P(R|B_3)P(B_3) \\ &= (0.75)\frac{1}{3} + (0.60)\frac{1}{3} + (0.45)\frac{1}{3} \\ &= 0.60 \end{aligned}$$

1.5 Law of Total Probability

Example 1.5.2

A company has three machines B_1 , B_2 , and B_3 making $1 \text{ k}\Omega$ resistors. Resistors within $50 \text{ }\Omega$ of the nominal value are considered acceptable. It has been observed that 80% of the resistors produced by B_1 and 90% of the resistors produced by B_2 are acceptable. The percentage for machine B_3 is 60%. Each hour, machine B_1 produces 3000 resistors, B_2 produces 4000 resistors, and B_3 produces 3000 resistors. All of the resistors are mixed together at random in one bin and packed for shipment. What is the probability that the company ships an acceptable resistor?

$A = \{\text{resistor is acceptable}\}$

$$P[A] = P[A|B_1] P[B_1] + P[A|B_2] P[B_2] + P[A|B_3] P[B_3]$$

1.5 Law of Total Probability

Example 1.5.2

$A = \{\text{resistor is acceptable}\}$

$$P[A] = P[A|B_1] P[B_1] + P[A|B_2] P[B_2] + P[A|B_3] P[B_3]$$

$$P[A|B_1] = 0.8 \qquad P[B_1] = 3000/10,000 = 0.3$$

$$P[A|B_2] = 0.9 \qquad P[B_2] = 0.4$$

$$P[A|B_3] = 0.6 \qquad P[B_3] = 0.3$$

$$\begin{aligned} P[A] &= P[A|B_1] P[B_1] + P[A|B_2] P[B_2] + P[A|B_3] P[B_3] \\ &= (0.8)(0.3) + (0.9)(0.4) + (0.6)(0.3) = 0.78 \end{aligned}$$

1.5 Law of Total Probability

Bayes' Rule

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \text{ when } P(B) > 0. \quad P(B|A) = ?$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \Rightarrow P(A \cap B) = P(A|B)P(B).$$

$$P(B|A) = \frac{P(B \cap A)}{P(A)}$$

For any two events A and B , where $P(A) \neq 0$, we have

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)}.$$

1.5 Law of Total Probability

Example 1.5.3

- The probability that a resistor is from machine B_3 is $P[B_3] = 0.3$.
- The probability that a resistor is *acceptable* — i.e., within $50\ \Omega$ of the nominal value — is $P[A] = 0.78$.
- Given that a resistor is from machine B_3 , the conditional probability that it is acceptable is $P[A|B_3] = 0.6$.

What is the probability that an acceptable resistor comes from machine B_3 ?

1.5 Law of Total Probability

Example 1.5.3

$$P[B_3|A] = \frac{P[A|B_3] P[B_3]}{P[A]}$$

$$P[B_3|A] = (0.6)(0.3)/(0.78) = 0.23$$

Similarly we obtain $P[B_1|A] = 0.31$ and $P[B_2|A] = 0.46$. Of all resistors within 50Ω of the nominal value, only 23% come from machine B_3 (even though this machine produces 30% of all resistors). Machine B_1 produces 31% of the resistors that meet the 50Ω criterion and machine B_2 produces 46% of them.

1.5 Law of Total Probability

Example 1.5.4

An insurance company has the following information about its customers:

- All customers insure at least one car.
 - 64% of the customers insure more than one car.
 - 20% of the customers insure a sports car.
 - Of those customers who insure more than one car, 15% insure a sports car.
1. Given a customer insures a sports car, what is the probability that he insures more than one car?
 2. What is the probability that a randomly selected customer insures exactly one car, and that car is not a sports car?

1.5 Law of Total Probability

Example 1.5.4

- 64% of the customers insure more than one car. $P(A) = 0.64$
- 20% of the customers insure a sports car. $P(B) = 0.20$
- Of those customers who insure more than one car, 15% insure a sports car.
 $P(B|A) = 0.15$

1. Given a customer insures a sports car, what is the probability that he insures more than one car?

$$P(A|B) = ?$$

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)} = \frac{0.15 * 0.64}{0.2} = 0.48$$

1.5 Law of Total Probability

Example 1.5.4

- 64% of the customers insure more than one car. $P(A) = 0.64$
- 20% of the customers insure a sports car. $P(B) = 0.20$
- Of those customers who insure more than one car, 15% insure a sports car.
 $P(B|A) = 0.15$

2. What is the probability that a randomly selected customer insures exactly one car, and that car is not a sports car?

$$P(A^c \cap B^c) = P(A \cup B)^c = 1 - P(A \cup B)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(B|A) = \frac{P(A \cap B)}{P(A)} \rightarrow P(A \cap B) = P(B|A)P(A) = 0.15 * 0.64 = 0.096$$

$$P(A \cup B) = 0.64 + 0.2 - 0.096 = 0.744 \rightarrow P(A^c \cap B^c) = 1 - 0.744 = 0.256$$

1.6 Independence

Introduction

I toss a fair coin A . Let B be the event that it rains tomorrow.

$$P(A|B) = ?, \quad P(A|B) = P(A)$$

Two events A and B are independent.

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = P(A) \quad \rightarrow P(A \cap B) = P(A)P(B).$$

1.6 Independence

Definition

Two events A and B are independent if the fact that A occurs does not affect the probability of B occurring. This means

$$P(A|B) = P(A), \text{ equivalently } P(A \cap B) = P(A)P(B).$$

$$\Rightarrow P(A|B)P(B) = P(A)P(B) \Rightarrow P(A \cap B) = P(A)P(B).$$

1.6 Independence

Example 1.6.1

Suppose that we flip two independent, fair coins. Consider the following two events:

A = the coins match (both are heads or both are tails)

B = the first coin is heads

Are these independent events?

$$S = \{HH, HT, TH, TT\}$$

$$A = \{HH, TT\} \rightarrow P(A) = \frac{2}{4} = \frac{1}{2}$$

$$B = \{HH, HT\} \rightarrow P(B) = \frac{2}{4} = \frac{1}{2}$$

$$(A \cap B) = \{HH\} \rightarrow P(A \cap B) = \frac{1}{4} = P(A)P(B)$$

A and B are independent

1.6 Independence

Definition

Warning!

Disjoint (mutually exclusive) $\neq$ Independent

Disjoint: $A \cap B = \emptyset$, $P(A \cup B) = P(A) + P(B)$

Independent: $P(A \cap B) = P(A)P(B)$, $P(A|B) = P(A)$, $P(B|A) = P(B)$

Suppose A and B are **disjoint**:

$$\text{If } P(B) \neq 0, \quad P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0}{P(B)} = 0.$$

If $P(A) \neq 0, P(B) \neq 0$ & disjoint $\Rightarrow$ Not independent.

1.6 Independence

Remark:

1) $P(A|B) = P(A) \Rightarrow P(B|A) = P(B), \quad (P(A), P(B) \neq 0)$

2) If A & B are independent, then

a) A & B^c are independent.

b) A^c & B are independent.

c) A^c & B^c are independent.

Prove (A)

$$P(A \cap B^c) = P(A)P(B^c)$$

$$P(A \cap B^c) = P(A - B) = P(A) - P(A \cap B)$$

$$P(A \cap B^c) = P(A) - P(A)P(B)$$

$$P(A \cap B^c) = P(A)(1 - P(B))$$

$$P(A \cap B^c) = P(A)P(B^c)$$

1.6 Independence

Remark:

3) $P(A_1 \cup A_2) = 1 - (1 - P(A_1))(1 - P(A_2))$.

Bonus Question (to be delivered until 07.10.2020 10:00 AM): Prove 3

1.6 Independence

Example 1.6.2

I toss a coin repeatedly until I observe a tails. Let X be the total number of coin tosses. Find:

a) $P(X=3)$

b) $P(X<5)$

a) $X = \{HHT\}$

$$P(X = 3) \rightarrow P(HHT) = P(H)P(H)P(T) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$$

b) $P(X < 5) \rightarrow (X = 1) \text{ or } (X = 2) \text{ or } (X = 3) \text{ or } (X = 4)$

$$\begin{aligned} P(X < 5) &\rightarrow P((X = 1) \cup (X = 2) \cup (X = 3) \cup (X = 4)) \\ &= P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4) = \\ &= 1/2 + 1/4 + 1/8 + 1/16 = 15/16 \end{aligned}$$

1.6 Independence

Three Independent Events

Three events A , B and C are **independent** if all of the following conditions hold

$$P(A \cap B) = P(A)P(B),$$

$$P(A \cap C) = P(A)P(C),$$

$$P(B \cap C) = P(B)P(C),$$

$$P(A \cap B \cap C) = P(A)P(B)P(C).$$

1.6 Independence

Example 1.6.2

Suppose that we flip three fair, mutually-independent coins. Define the following events:

- A_1 is the event that coin 1 matches coin 2. $A_1 = \{HHH, HHT, TTH, TTT\}$
- A_2 is the event that coin 2 matches coin 3. $A_2 = \{HHH, THH, HTT, TTT\}$
- A_3 is the event that coin 3 matches coin 1. $A_3 = \{HHH, HTH, THT, TTT\}$

Are A_1, A_2, A_3 mutually independent?

Every outcome has probability $(1/2)^3 = 1/8$

$$\rightarrow P(A_1) = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2} \qquad \rightarrow P(A_2) = \frac{1}{2} \qquad \rightarrow P(A_3) = \frac{1}{2}$$

$$\Pr(A_1 \cap A_2) = \Pr(HHH) + \Pr(TTT) = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$$

$$\Pr(A_1) \Pr(A_2) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

A_1 and A_2 are independent

1.6 Independence

Example 1.6.2

- A_1 is the event that coin 1 matches coin 2. $A_1 = \{HHH, HHT, TTH, TTT\}$
- A_2 is the event that coin 2 matches coin 3. $A_2 = \{HHH, THH, HTT, TTT\}$
- A_3 is the event that coin 3 matches coin 1. $A_3 = \{HHH, HTH, THT, TTT\}$

Are A_1, A_2, A_3 mutually independent?

We can easily prove that:

$$\Pr(A_1 \cap A_3) = \Pr(A_1) \cdot \Pr(A_3) \text{ and } \Pr(A_2 \cap A_3) = \Pr(A_2) \cdot \Pr(A_3)$$

$$\begin{aligned} \Pr(A_1 \cap A_2 \cap A_3) &= \Pr(HHH) + \Pr(TTT) = \frac{1}{8} + \frac{1}{8} \\ &= \frac{1}{4} \end{aligned}$$

pairwise independent

The three events A_1, A_2 , and A_3 are not mutually independent $\leftarrow \neq \Pr(A_1) \Pr(A_2) \Pr(A_3) = \frac{1}{8}$

1.6 Independence

Example 1.6.2

Monitor two consecutive packets going through a router. Classify each one as video (v) if it was sent from a Youtube server or as ordinary data (d) otherwise. Your observation is a sequence of two letters (either v or d). For example, two video packets corresponds to vv . The two packets are independent and the probability that any one of them is a video packet is 0.8. Denote the identity of packet i by C_i . If packet i is a video packet, then $C_i = v$; otherwise, $C_i = d$. Count the number N_V of video packets in the two packets you have observed. Determine whether the following pairs of events are independent:

(a) $\{N_V = 2\}$ and $\{N_V \geq 1\}$

$$P[\{vv\}] = (0.8)^2 = 0.64$$

(b) $\{N_V \geq 1\}$ and $\{C_1 = v\}$

$$P[\{vd\}] = (0.8)(0.2) = 0.16$$

(c) $\{C_2 = v\}$ and $\{C_1 = d\}$

$$P[\{dv\}] = (0.2)(0.8) = 0.16$$

(d) $\{C_2 = v\}$ and $\{N_V \text{ is even}\}$

$$P[\{dd\}] = (0.2)^2 = 0.04$$

1.6 Independence

Example 1.6.2

(a) $\{N_V = 2\}$ and $\{N_V \geq 1\}$

$$P[N_V = 2, N_V \geq 1] = P[N_V = 2] = P[\{vv\}] = 0.64$$

$$P[N_V \geq 1] = P[\{vd, dv, vv\}] = 0.96$$

$$P[N_V = 2] P[N_V \geq 1] = (0.64)(0.96) \neq P[N_V = 2, N_V \geq 1]$$

the two events are dependent

1.6 Independence

Example 1.6.2

(b) $\{N_V \geq 1\}$ and $\{C_1 = v\}$

$$P[N_V \geq 1, C_1 = v] = P[\{vd, vv\}] = 0.80$$

$$P[N_V \geq 1] = 0.96$$

$$P[C_1 = v] = 0.8$$

$$P[N_V \geq 1] P[C_1 = v] = (0.96)(0.8) = 0.768 \neq P[N_V \geq 1, C_1 = v]$$

the two events are dependent

1.6 Independence

Example 1.6.2

(c) $\{C_2 = v\}$ and $\{C_1 = d\}$

$$P[C_1 = d, C_2 = v] = P[\{dv\}] = 0.16$$

$$P[C_1 = d] P[C_2 = v] = (0.2)(0.8) = 0.16$$

the two events are independent

1.6 Independence

Example 1.6.2

(d) $\{C_2 = v\}$ and $\{N_V \text{ is even}\}$

$$P[C_2 = v, N_V \text{ is even}] = P[\{vv\}] = 0.64$$

$$P[C_2 = v] = P[\{dv, vv\}] = 0.8$$

$$P[N_V \text{ is even}] = P[\{dd, vv\}] = 0.68$$

$$\begin{aligned} P[C_2 = v] P[N_V \text{ is even}] &= (0.8)(0.68) \\ &= 0.544 \neq P[C_2 = v, N_V \text{ is even}] \end{aligned}$$

the two events are dependent

Chapter 1: Selected Problems

Textbook:

Roy D. Yates and David J. Goodman, *Probability and Stochastic Processes: A Friendly Introduction For Electrical and Computer Engineers*, third edition, Wiley, 2014.

1.2.1, 1.2.2, 1.3.1, 1.3.4, 1.3.5, 1.4.1, 1.4.2, 1.4.3, 1.5.2, 1.6.1, 1.6.5, 1.6.6, 1.6.7, 1.6.8, 1.6.9