

2.2 Counting Methods

Introduction

For a finite sample space S with equally likely outcomes, the probability of an event A is given by

$$P(A) = \frac{|A|}{|S|} = \frac{M}{N}$$

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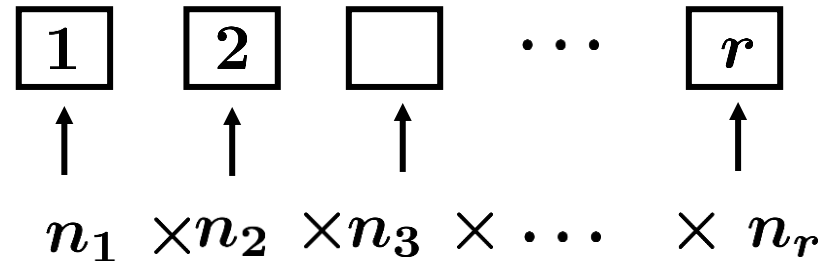
Suppose that I want to purchase a tablet computer. I can choose either a large or a small screen; a 64GB, 128GB, or 256GB storage capacity, and a black or white cover. How many different options do I have?

$$2 \times 3 \times 2 = 12$$

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Multiplication Principle

If we are to perform r experiments in order such that there are n_1 possible outcomes of the first experiment, n_2 possible outcomes of the second experiment, $\dots$, n_r possible outcomes of the r^{th} experiment, then there is a total of $n_1 \times n_2 \times n_3 \times \dots \times n_r$ outcomes of the sequence of the r experiments.



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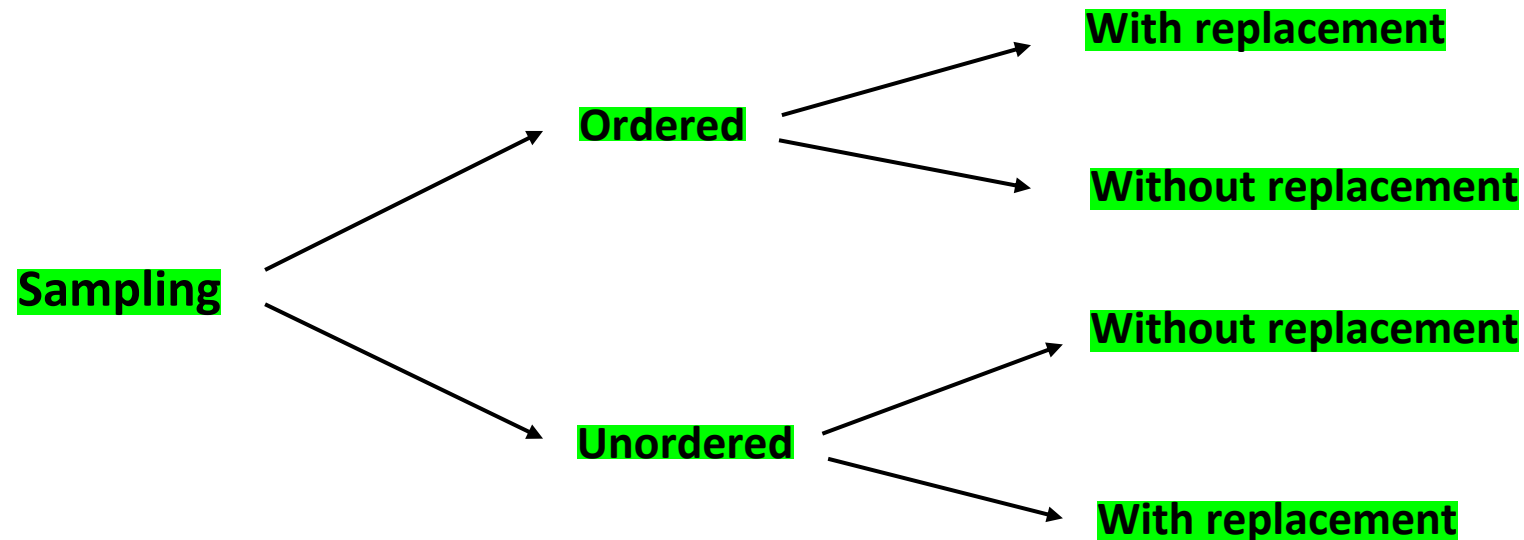
Terminology

- Drawing (choosing) objects from a set $A = \{a_1, a_2, \dots, a_n\}$ is referred to as sampling.
- We will often draw multiple samples from a set. If we put the object back after each draw, this is called **sampling with replacement**; if not it is called **sampling without replacement**.
- The result of drawing multiple samples can be **ordered** (order of draws **matters**; $1, 2, 3 \neq 2, 3, 1$) or **unordered** ($1, 2, 3 = 2, 3, 1$).

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General scenario

We have a set of n elements, e.g., $A = \{1, 2, \dots, n\}$ and we draw k samples from the set:



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Ordered Sampling with Replacement (repetition allowed)

For $A = \{1, 2, 3\}$, $k = 2$

$(1, 1)$ $(2, 1)$ $(3, 1)$
 $(1, 2)$ $(2, 2)$ $(3, 2)$ $\longrightarrow$ 9 Possibilities
 $(1, 3)$ $(2, 3)$ $(3, 3)$

In general: $A = \{1, 2, \dots, n\}$

$\boxed{1}$ $\boxed{2}$ $\boxed{}$ $\dots$ $\boxed{k}$
 $\uparrow$ $\uparrow$ $\uparrow$ $$ $\uparrow$
 $n \times n \times n \times \dots \times n \quad \Rightarrow \quad n^k$

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Ordered Sampling without Replacement (repetition not allowed)

$$A = \{1, 2, 3\}, k = 2$$

$$\begin{array}{l} (1, 2) \quad (2, 1) \quad (3, 1) \\ (1, 3) \quad (2, 3) \quad (3, 2) \end{array} \longrightarrow 6 \text{ Possibilities}$$

In general: $A = \{1, 2, \dots, n\}$

$$\begin{array}{ccccccc} \square & \square & \square & \dots & \square \\ \uparrow & \uparrow & \uparrow & & \uparrow \\ n \times & (n-1) \times & (n-2) \times & \dots & \times (n-k+1) \end{array}$$

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Ordered Sampling without Replacement (repetition not allowed)

Number of k -permutations of n -objects:

$$\underline{P_k^n} = n \times (n - 1) \times \dots \times (n - k + 1) = \frac{n!}{(n - k)!}.$$

The number of k -permutations of n distinguishable objects is given by

$$P_k^n = \frac{n!}{(n - k)!}, \text{ for } 0 \leq k \leq n.$$

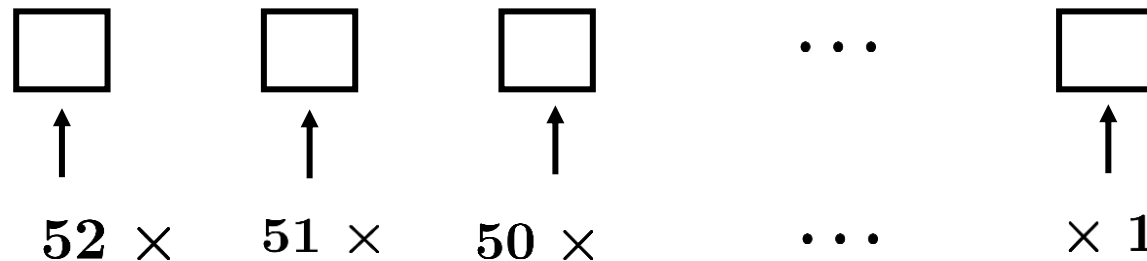
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Ordered Sampling without Replacement (repetition not allowed)

Example 2.2.1

Shuffle a deck of 52 cards. How many outcomes are possible? (In other words, how many different ways can you order 52 distinct cards? How many different permutations of 52 distinct cards exist?)

Answer: 52!



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Ordered Sampling without Replacement (repetition not allowed)

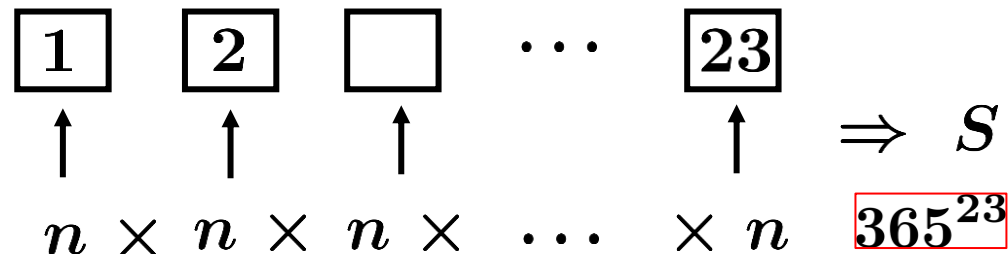
Example 2.2.2: Birthday Paradox:

What is the chance of 23 people in a room to have at least two people having the same birthday. And for 75 what is the chance of at least two people matching?

$P(A)$: Probability at least two people having the same birthday

$P(A^c)$: Probability that no matches

$$P(A) = 1 - P(A^c) = 1 - \frac{|A^c|}{|S|} = 1 - \frac{P_{23}^{365}}{365^{23}} = 0.507$$



for 75 people:

$$1 - \frac{P_{75}^{365}}{365^{75}} = 0.999$$

