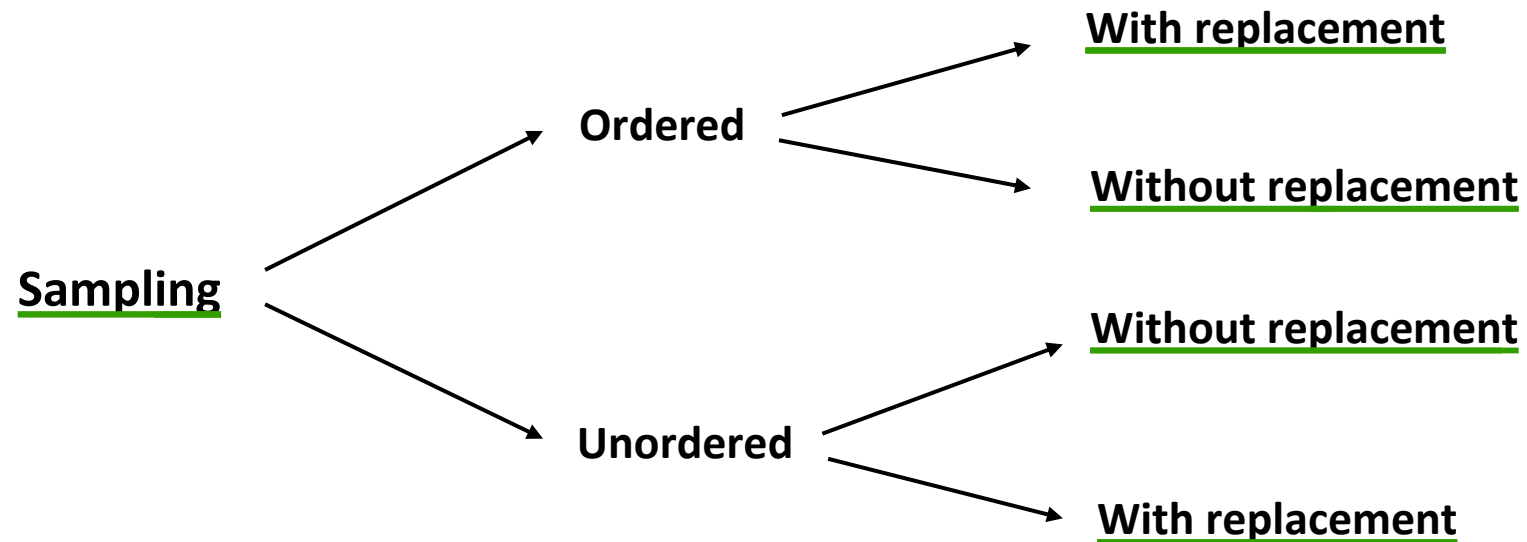


2.2 Counting Methods

General scenario

We have a set of n elements, e.g., $A = \{1, 2, \dots, n\}$ and we draw k samples from the set:



2.2 Counting Methods

Unordered Sampling without Replacement (Combinations):

There are n distinguishable objects; we want to choose k objects, but ordering does not matter: $1, 2, 3 = 2, 3, 1 = 3, 2, 1$.

Let $A = \{1, 2, 3\}$ and $k = 2$, then

$\{1, 2\} \quad \{1, 3\} \quad \{2, 3\} \longrightarrow 3$ possibilities

In general:

$\binom{n}{k}$: # of ways to choose k elements from n elements (Unordered): k -

Combinations

If ordered: $P_k^n = \frac{n!}{(n-k)!} = k! \binom{n}{k}$. If unordered: $\binom{n}{k} = \frac{P_k^n}{k!} = \frac{n!}{k!(n-k)!}$.

2.2 Counting Methods

Unordered Sampling without Replacement (Combinations):

Thus the number of k -combinations of n objects is:

$$\binom{n}{k} = \frac{n!}{k!(n - k)!}.$$

The number of ways to choose k objects out of n distinguishable objects is equal to $\binom{n}{k}$.

2.2 Counting Methods

Unordered Sampling without Replacement (Combinations):

- The number of combinations of seven cards chosen from a deck of 52 cards is

$$\binom{52}{7} = \frac{52 \times 51 \times \cdots \times 46}{2 \times 3 \times \cdots \times 7} = 133,784,560,$$

- There are 11 players on a basketball team. The starting lineup consists of five players. There are $\binom{11}{5} = 462$ possible starting lineups.
- There are $\binom{120}{60} \approx 10^{36}$ ways of dividing 120 students enrolled in a probability course into two sections with 60 students in each section.
- There are four queens in a deck of 52 cards. You are given seven cards at random from the deck. What is the probability that you have no queens?

The sample space contains $H = \binom{52}{7}$ possible combinations of seven cards, each with probability $1/H$

There are $H_{NQ} = \binom{52-4}{7}$ combinations with no queens

$$H_{NQ}/H = 0.5504$$

2.2 Counting Methods

Unordered Sampling without Replacement (Combinations):

Example 2.2.3

I choose 3 cards from the standard deck of cards. What is the probability that these cards contain at least one ace? A is the event that at least one ace is selected.

$$|S| = \binom{52}{3} \quad \leftarrow \text{Sample space}$$

$$|A^c| = \binom{48}{3}$$

$$P(A) = 1 - \frac{\binom{48}{3}}{\binom{52}{3}}$$

2.2 Counting Methods

Multinomial Coefficients:

More generally if $n = n_1 + n_2 + \dots + n_r$, we define

$$\binom{n}{n_1, n_2, \dots, n_r} = \frac{n!}{n_1! n_2! \dots n_r!}.$$

 $\binom{n}{n_1, n_2, \dots, n_r}$ is the number of possible divisions of n distinct objects into r distinct groups of respective sizes $n_1, n_2, \dots, n_r$.

2.2 Counting Methods

Unordered Sampling without Replacement (Combinations):

Example: the professor uses a *curve* in determining student grades. When there are ten students in a probability class, the professor always issues two grades of A, three grades of B, three grades of C and two grades of F. How many different ways can the professor assign grades to the ten students?

$4^{10} = 1,048,576$ ways of assigning grades to ten students.

we are limited to choosing $n_0 = 2$ students to receive an *A*, $n_1 = 3$ students to receive a *B*, $n_2 = 3$ students to receive a *C* and $n_3 = 2$ students to receive an *F*. The number of ways that fit the curve is the multinomial coefficient

$$\binom{n}{n_0, n_1, n_2, n_3} = \binom{10}{2, 3, 3, 2} = \frac{10!}{2!3!3!2!} = 25,200.$$

2.2 Counting Methods

Bernoulli Trials:

Example 2.2.4 We toss an unfair coin ($P(H) = p$) 5 times. What is the probability of observing ($k = 1, 3$) heads?

$$k = 1$$

$$A = \{HTTTT, THTTT, TTHTT, TTTHT, TTTTH\}$$


$$p(1 - p)^4$$

$$P(A) = 5p(1 - p)^4$$

$$k = 3$$

$$A = \{HHHTT, HHHTH, \dots\}$$


$$p^3(1 - p)^2$$

$$P(A) = |A| \cdot p^3(1 - p)^2 \quad |A| = \binom{5}{3}$$

2.2 Counting Methods

Binomial Formula

Bernoulli trial (or binomial trial): is a random experiment with exactly two possible outcomes, "success" and "failure", in which the probability of success is the same every time the experiment is conducted.

For n independent Bernoulli trials where each trial has success probability p , the probability of k successes is given by

$$P(k) = \binom{n}{k} p^k (1 - p)^{n-k}.$$

2.2 Counting Methods

Unordered Sampling with Replacement (repetition allowed):

Example: $A = \{1, 2, 3\}$, $n = 3$, $k = 2$

(1, 1) (2, 2)

(1, 2) (2, 3) $\longrightarrow$ 6 Cases.

(1, 3) (3, 3)

The total number of distinct k samples from an n -element set such that repetition is allowed and ordering does not matter is

$$\binom{n - k + 1}{k} = \binom{n - k + 1}{n - 1}.$$

2.2 Counting Methods

Review

Let's summarize the formulas for the four categories of sampling.

<u>ordered sampling with replacement</u>	n^k
<u>ordered sampling without replacement</u>	$P_k^n = \frac{n!}{(n-k)!}$
<u>unordered sampling without replacement</u>	$\binom{n}{k} = \frac{n!}{k!(n-k)!}$
<u>unordered sampling with replacement</u>	$\binom{n+k-1}{k}$

2.2 Counting Methods

Example 2.2.5

Consider a binary code with 4 bits (0 or 1) in each code word. An example of a code word is 0110.

- (a) How many different code words are there? $2 \times 2 \times 2 \times 2 = 2^4 = 16$
- (b) How many code words have exactly two zeroes? $\binom{4}{2} = 6$
- (c) How many code words begin with a zero? $1 \times 2 \times 2 \times 2 = 8$
- (d) In a constant-ratio binary code, each code word has N bits. In every word, M of the N bits are 1 and the other $N - M$ bits are 0. How many different code words are in the code with $N = 8$ and $M = 3$?

$$\binom{8}{3} = 56 \text{ code words.}$$