

EELE 3340

Probability Theory



Chapter 3

Discrete Random Variables

Dr. Mohammed El-Absi

3.1 Definition

Introduction

Example. Toss a coin five times. Observe the number of heads:

$$S = \{TTTTT, TTTTH, \dots, HHHHH\}.$$

$$X = 0, 1, 2, 3, 4, \text{ or } 5.$$

We define a **random variable** that gets its value from the outcome of the random experiment:

Definition: A random variable is a real-valued variable that gets its value from a random experiment.

3.1 Definition

Introduction

Formal Definition: A random variable is a real-valued function on the sample space:

$$X : S \rightarrow \mathbb{R}, X(\{HHTHT\}) = 3.$$

Definition: Range of X is the set of possible values for X .

In the above example, $\text{Range}(X) = S_X = \{0, 1, 2, 3, 4, 5\}$.

We show random variables with capital letters X, Y, Z .

3.1 Definition

Examples:

Example. Flip a coin twice, X = the number of heads

$$\text{Range}(X) = R = \{0, 1, 2\}.$$

Example. T: Lifetime of a certain product:

$$\text{Range} = \{X : X \in \mathbb{R}; x \geq 0\} = \mathbb{R}^+ = [0, \infty).$$

3.1 Definition

Examples:

The experiment is to test six integrated circuits and after each test observe whether the circuit is accepted (*a*) or rejected (*r*). Each observation is a sequence of six letters where each letter is either *a* or *r*.

The sample space S consists of the 64 possible sequences.

A random variable related to this experiment is N , the number of accepted circuits. For outcome s_8 , $N = 5$ circuits are accepted. The range of N is

$$S_N = \{0, 1, \dots, 6\}$$

3.1 Definition

Introduction

Countable set:

- a) Finite set
- b) One-to-one correspondence with Natural Numbers

$$\mathbb{N} = \{1, 2, 3, 4, \dots\}$$

$$\text{i.e., } R = \{a_1, a_2, a_3, \dots\}$$

i.e., I can “**list**” the elements.

3.1 Definition

Introduction

Countably infinite sets:

$$\mathbb{N} = \{1, 2, 3, 4, \dots\}$$

$$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$$

List: $\mathbb{Z} = \{0, -1, 1, -2, 2, -3, 3, \dots\}$

$$\mathbb{Q} = \left\{ \frac{a}{b} : a, b \in \mathbb{Z} \right\} \quad \text{Countable}$$

However $\mathbb{R}$ is Not countable, in fact

$[0, 1] = \{x \in \mathbb{R}, 0 \leq x \leq 1\}$ **is Not countable.**

3.1 Definition

Discrete Random Variable

Definition: X is a **discrete random** variable, if its range is countable.

$$S_X = \{x_1, x_2, x_3, \dots\}.$$

We show the values in the range by lower case letters.

3.2 Probability Mass Function

Example 3.2.1

Toss a fair coin twice, $X = \#$ of heads.

Find the range of X , R_X .

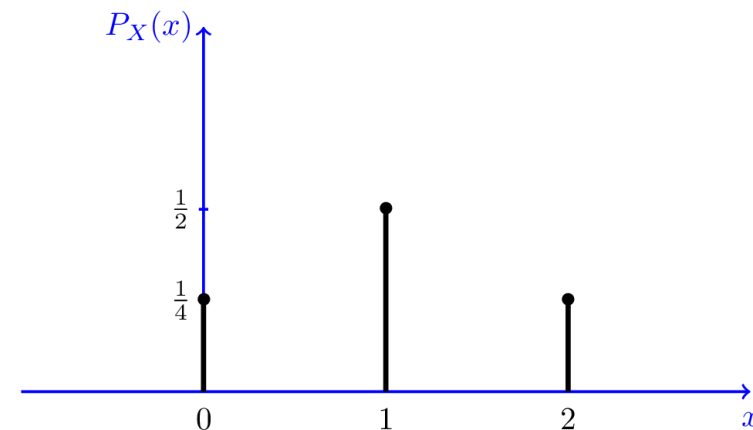
$$S = \{HH, HT, TH, TT\}$$

$$S_X = \{2, 1, 0\}$$

$$P_X(0) = P(X = 0) = P(TT) = \frac{1}{4}$$

$$P_X(1) = P(X = 1) = P(\{HT, TH\}) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

$$P_X(2) = P(X = 2) = P(HH) = \frac{1}{4}$$



3.2 Probability Mass Function

Definition

X is a discrete random variable,

$$\text{Range}(X) = S_X = \{x_1, x_2, x_3, \dots\}.$$

The function:

$$P_X(x_k) = P(X = x_k), \text{ for } k = 1, 2, 3, \dots,$$

is called the **probability mass function (PMF)** of X .

3.2 Probability Mass Function

Example 3.2.2

I have an unfair coin for which $P(H) = p$, where $0 < p < 1$. I toss the coin repeatedly until I observe a heads for the first time. Let Y be the total number of coin tosses. Find the distribution of Y .

$$R_Y = \mathbb{N} = \{1, 2, 3, \dots\}$$

$$P_Y(1) = P(Y = 1) = P(H) = p,$$

$$P_Y(y) = \begin{cases} (1-p)^{y-1}p & \text{for } y = 1, 2, 3, \dots \\ 0 & \text{otherwise} \end{cases}$$

$$P_Y(2) = P(Y = 2) = P(TH) = (1-p)p,$$

$$P_Y(3) = P(Y = 3) = P(TTH) = (1-p)^2p,$$

$$P_Y(k) = P(Y = k) = P(TT\dots TH) = (1-p)^{k-1}p.$$

3.2 Probability Mass Function

Theorem

For a discrete random variable with PMF $P_X(x)$ and Range

$$R_X = S_X = \{x_1, x_2, x_3, \dots\}$$

- a) $0 \leq P_X(x_k) \leq 1$ for all $x_k \in S_X$.
- b) $\sum_{k=1}^{\infty} P_X(x_k) = 1$.
- c) $B \subset S_X, P(X \in B) = P(B) = \sum_{x_k \in B} \text{Prob}\{X = x_k\} = \sum_{x_k \in B} P_X(x_k)$

3.2 Probability Mass Function

Example 3.2.3

The random variable N has PMF

$$P_N(n) = \begin{cases} c/n & n = 1, 2, 3, \\ 0 & \text{otherwise.} \end{cases}$$

Find

(a) The value of the constant c

$$\sum_{n=1}^3 P_N(n) = c \left(1 + \frac{1}{2} + \frac{1}{3} \right) = 1. \longrightarrow c = 6/11$$

(b) $P[N = 1]$

$$P[N = 1] = P_N(1) = c = 6/11$$

(c) $P[N \geq 2]$

$$\begin{aligned} P[N \geq 2] &= P_N(2) + P_N(3) \\ &= c/2 + c/3 = 5/11. \end{aligned}$$

(d) $P[N > 3] = \sum_{n=4}^{\infty} P_N(n) = 0$