

3.3 Families of Discrete Random Variables

Example 3.3.1

Test one circuit and observe X , the number of rejects. What is $P_X(x)$ the PMF of random variable X ?

Because there are only two outcomes in the sample space, $X = 1$ with probability p and $X = 0$ with probability $1 - p$,

$$P_X(x) = \begin{cases} 1 - p & x = 0, \\ p & x = 1, \\ 0 & \text{otherwise.} \end{cases}$$

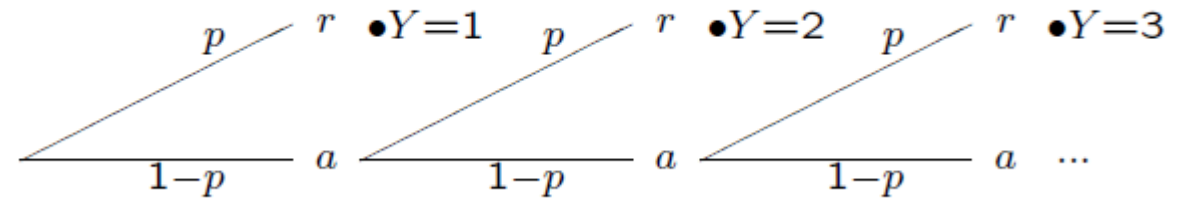
Therefore, the number of circuits rejected in one test is a Bernoulli (p) random variable.

3.3 Families of Discrete Random Variables

Example 3.3.2

In a sequence of independent tests of integrated circuits, each circuit is rejected with probability p . Let Y equal the number of tests up to and including the first test that results in a reject. What is the PMF of Y ?

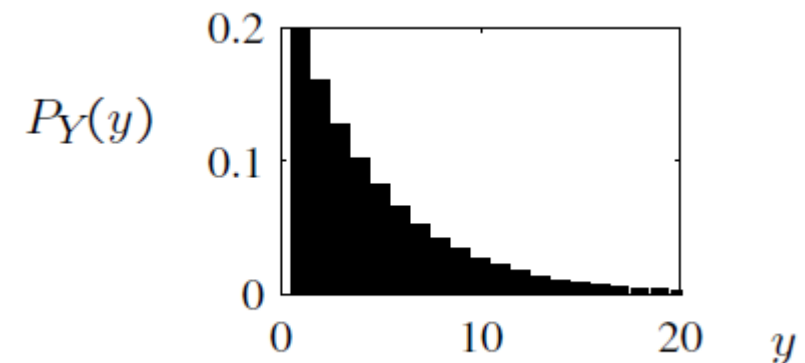
$$P_Y(y) = \begin{cases} p(1-p)^{y-1} & y = 1, 2, \dots \\ 0 & \text{otherwise.} \end{cases}$$



Y is referred to as a *geometric random variable* because the probabilities in the PMF constitute a geometric series.

If there is a 0.2 probability of a reject

$$P_Y(y) = \begin{cases} (0.2)(0.8)^{y-1} & y = 1, 2, \dots \\ 0 & \text{otherwise.} \end{cases}$$



3.3 Families of Discrete Random Variables

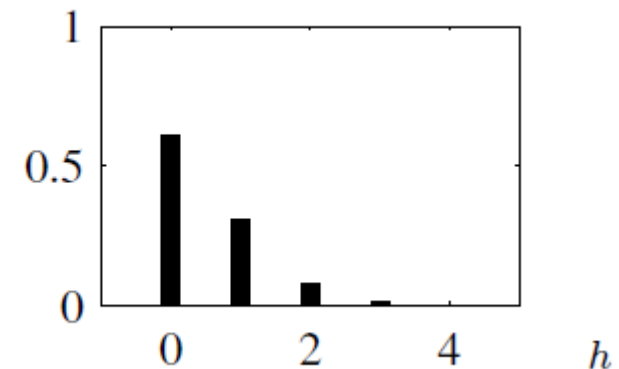
Example 3.3.3

The number of hits at a website in any time interval is a Poisson random variable. A particular site has on average $\lambda = 2$ hits per second. What is the probability that there are no hits in an interval of 0.25 seconds? What is the probability that there are no more than two hits in an interval of one second?

In an interval of 0.25 seconds, the number of hits H is a Poisson random variable with $\alpha = \lambda T = (2 \text{ hits/s}) \times (0.25 \text{ s}) = 0.5$ hits. The PMF of H is

$$P_H(h) = \begin{cases} 0.5^h e^{-0.5} / h! & h = 0, 1, 2, \dots \\ 0 & \text{otherwise.} \end{cases} \quad P_X(k) = \frac{e^{-\alpha} \alpha^k}{k!}, \quad k = 0, 1, 2, 3, \dots \quad P_H(h)$$

The probability of no hits is $P[H = 0] = P_H(0) = (0.5)^0 e^{-0.5} / 0! = 0.607$



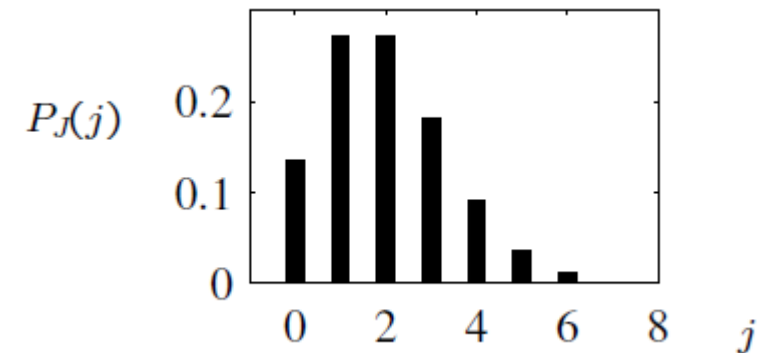
3.3 Families of Discrete Random Variables

Example 3.3.3

The number of hits at a website in any time interval is a Poisson random variable. A particular site has on average $\lambda = 2$ hits per second. What is the probability that there are no hits in an interval of 0.25 seconds? What is the probability that there are no more than two hits in an interval of one second?

In an interval of 1 second, $\alpha = \lambda T = (2 \text{ hits/s}) \times (1 \text{ s}) = 2$ hits.

$$P_J(j) = \begin{cases} 2^j e^{-2} / j! & j = 0, 1, 2, \dots \\ 0 & \text{otherwise.} \end{cases}$$



$$\{J \leq 2\} = \{J = 0\} \cup \{J = 1\} \cup \{J = 2\}$$

$$\begin{aligned} P[J \leq 2] &= P[J = 0] + P[J = 1] + P[J = 2] = P_J(0) + P_J(1) + P_J(2) \\ &= e^{-2} + 2^1 e^{-2} / 1! + 2^2 e^{-2} / 2! = 0.677. \end{aligned}$$

3.3 Families of Discrete Random Variables

Example 3.3.4

The number of database queries processed by a computer in any 10-second interval is a Poisson random variable, K , with $\alpha = 5$ queries. What is the probability that there will be no queries processed in a 10-second interval? What is the probability that at least two queries will be processed in a 2-second interval?

$$P_K(k) = \begin{cases} 5^k e^{-5} / k! & k = 0, 1, 2, \dots \\ 0 & \text{otherwise.} \end{cases} \quad P[K = 0] = P_K(0) = e^{-5} = 0.0067$$

$$\begin{aligned} \lambda &= 0.5 \\ \alpha &= 2\lambda = 1 \end{aligned} \quad P_N(n) = \begin{cases} e^{-1} / n! & n = 0, 1, 2, \dots \\ 0 & \text{otherwise.} \end{cases}$$

$$P[N \geq 2] = 1 - P_N(0) - P_N(1) = 1 - e^{-1} - e^{-1} = 0.264.$$

3.3 Families of Discrete Random Variables

Example 3.3.5

Each time a modem transmits one bit, the receiving modem analyzes the signal that arrives and decides whether the transmitted bit is 0 or 1. It makes an error with probability p , independent of whether any other bit is received correctly.

(a) If the transmission continues until the receiving modem makes its first error, what is the PMF of X , the number of bits transmitted?

$$P_X(x) = \begin{cases} p(1-p)^{x-1} & x = 1, 2, \dots \\ 0 & \text{otherwise.} \end{cases}$$

(b) If $p = 0.1$, what is the probability that $X = 10$? What is the probability that $X \geq 10$?

$$P_X(10) = (0.1)(0.9)^9 = 0.0387$$

The probability that at least 10 bits are sent is $P[X \geq 10] = \sum_{x=10}^{\infty} P_X(x)$

3.3 Families of Discrete Random Variables

Example 3.3.5

(c) If the modem transmits 100 bits, what is the PMF of Y , the number of errors?

Y is the number of successes in 100 independent trials

$$P_Y(y) = \binom{100}{y} p^y (1-p)^{100-y}$$

(d) If $p = 0.01$ and the modem transmits 100 bits, what is the probability of $Y = 2$ errors at the receiver? What is the probability that $Y \leq 2$?

$$P_Y(2) = \binom{100}{2} (0.01)^2 (0.99)^{98} = 0.1849.$$

$$P[Y \leq 2] = P_Y(0) + P_Y(1) + P_Y(2)$$

$$= (0.99)^{100} + 100(0.01)(0.99)^{99} + \binom{100}{2} (0.01)^2 (0.99)^{98} = 0.9207$$

3.3 Families of Discrete Random Variables

Example 3.3.5

(e) If the transmission continues until the receiving modem makes three errors, what is the PMF of Z , the number of bits transmitted?

Thus Z has the Pascal PMF

$$P_Z(z) = \binom{z-1}{2} p^3 (1-p)^{z-3}$$

(f) If $p = 0.25$, what is the probability of $Z = 12$ bits transmitted until the modem makes three errors?

$$P_Z(12) = \binom{11}{2} (0.25)^3 (0.75)^9 = 0.0645$$

3.3 Families of Discrete Random Variables

Example 3.3.6

Let X and Y be two independent discrete random variables with the following PMFs:

$$P_X(k) = \begin{cases} \frac{1}{4} & \text{for } k = 1 \\ \frac{1}{8} & \text{for } k = 2 \\ \frac{1}{8} & \text{for } k = 3 \\ \frac{1}{2} & \text{for } k = 4 \\ 0 & \text{otherwise} \end{cases}$$

$$P_Y(k) = \begin{cases} \frac{1}{6} & \text{for } k = 1 \\ \frac{1}{6} & \text{for } k = 2 \\ \frac{1}{3} & \text{for } k = 3 \\ \frac{1}{3} & \text{for } k = 4 \\ 0 & \text{otherwise} \end{cases}$$

a. Find $P(X \leq 2 \text{ and } Y \leq 2)$.

(a) X and Y are two independent random variables. So:

$$\begin{aligned} P(X \leq 2 \text{ and } Y \leq 2) &= P(X \leq 2) \cdot P(Y \leq 2) \\ &= (P_X(1) + P_X(2)) \cdot (P_Y(1) + P_Y(2)) \\ &= \left(\frac{1}{4} + \frac{1}{8}\right) \left(\frac{1}{6} + \frac{1}{6}\right) = \frac{3}{8} \cdot \frac{1}{3} = \frac{1}{8} \end{aligned}$$

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Example 3.3.6

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b. Find $P(X > 2 \text{ or } Y > 2)$

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= P(X > 2) + P(Y > 2) - P(X > 2 \text{ and } Y > 2) \\ &= \left(\frac{1}{8} + \frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{3}\right) - \left(\frac{1}{8} + \frac{1}{2}\right) \cdot \left(\frac{1}{3} + \frac{1}{3}\right) \\ &= \frac{5}{8} + \frac{2}{3} - \frac{5}{8} \cdot \frac{2}{3} = \frac{21}{24} = \frac{7}{8} \end{aligned}$$

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$$P_Y(k) = \begin{cases} \frac{1}{6} & \text{for } k = 1 \\ \frac{1}{6} & \text{for } k = 2 \\ \frac{1}{3} & \text{for } k = 3 \\ \frac{1}{3} & \text{for } k = 4 \\ 0 & \text{otherwise} \end{cases}$$

c. Find $P(X > 2|Y > 2)$.

$$\begin{aligned} P(X > 2|Y > 2) &= P(X > 2) \\ &= \frac{1}{8} + \frac{1}{2} = \frac{5}{8} \end{aligned}$$

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d. Find $P(X < Y)$.

$$\begin{aligned} P(X < Y) &= \sum_{k=1}^4 P(X < Y | Y = k) \cdot P(Y = k) \\ &= P(X < 1 | Y = 1) \cdot P(Y = 1) + P(X < 2 | Y = 2) \cdot P(Y = 2) \\ &\quad + P(X < 3 | Y = 3) \cdot P(Y = 3) + P(X < 4 | Y = 4) \cdot P(Y = 4) \\ &= 0 \cdot \frac{1}{6} + \frac{1}{4} \cdot \frac{1}{6} + \left(\frac{1}{4} + \frac{1}{8}\right) \cdot \frac{1}{3} + \left(\frac{1}{4} + \frac{1}{8} + \frac{1}{8}\right) \cdot \frac{1}{3} = \frac{1}{24} + \frac{1}{8} + \frac{1}{6} = \frac{10}{24} = \frac{5}{12} \end{aligned}$$

Selected Problems

3.2.2, 3.2.3, 3.2.5, 3.2.7, **3.2.8**, 3.3.1, 3.3.2, 3.3.3, 3.3.7, 3.3.8, 3.3.10, 3.3.11,
3.3.14, 3.3.15,