

3.5 Averages and Expected Value

Example 3.5.1

For one quiz, 10 students have the following grades (on a scale of 0 to 10):

9, 5, 10, 8, 4, 7, 5, 5, 8, 7

Find the mean, the median, and the mode.

The mean value is $68/10 = 6.8$

median is 7

The mode is 5

3.5 Averages and Expected Value

Definitions

Expected value (= mean=average):

$$a_1, a_2, \dots, a_n \Rightarrow \bar{a} = \frac{a_1 + a_2 + \dots + a_n}{n}.$$

Definition. Let X be a discrete random variable with range $S_X = \{x_1, x_2, x_3, \dots\}$. The **expected value** of X , denoted by $E[X]$ is defined as

$$E[X] = \mu_X = \sum_{x_k \in S_X} x_k P(X = x_k) = \sum_{x_k \in S_X} x_k P_X(x_k).$$

3.5 Averages and Expected Value

Definitions

A mode of random variable X is a number x_{mod} satisfying $P_X(x_{\text{mod}}) \geq P_X(x)$ for all x .

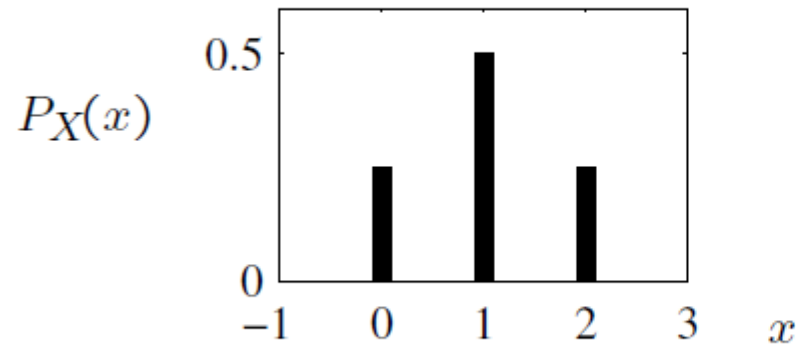
A median, x_{med} , of random variable X is a number that satisfies

$$P[X \leq x_{\text{med}}] \geq 1/2, \quad P[X \geq x_{\text{med}}] \geq 1/2.$$

3.5 Averages and Expected Value

Example 3.5.2

Random variable X has PMF



$$P_X(x) = \begin{cases} 1/4 & x = 0, \\ 1/2 & x = 1, \\ 1/4 & x = 2, \\ 0 & \text{otherwise.} \end{cases}$$

What is $E[X]$?

$$E[X] = \mu_X = 0 \cdot P_X(0) + 1 \cdot P_X(1) + 2 \cdot P_X(2)$$

$$= 0(1/4) + 1(1/2) + 2(1/4) = 1$$

3.5 Averages and Expected Value

Example 3.5.3

A probability function, $P(x)$, is defined by the following table:

x	0	1	2	3	4	5	6
$P(x)$	0.17	0.29	0.27	0.16	0.07	0.03	0.01

Determine (a) the mean, (b) the median, (c) the mode and

(a) Mean = 1.8

(b) median = 2

(c) mode = 1

3.5 Averages and Expected Value Families

Let $X \sim \text{Bernoulli}(p)$, **find** $E[X]$.

$$E[X] = 0 \cdot P_X(0) + 1P_X(1) = 0(1 - p) + 1(p) = p$$

Let $X \sim \text{Geometric}(p)$, **find** $E[X]$.

$$P_X(x) = \begin{cases} pq^{x-1} & x = 1, 2, \dots \\ 0 & \text{otherwise.} \end{cases} \quad q = 1 - p$$

$$E[X] = \sum_{x=1}^{\infty} xP_X(x) = \sum_{x=1}^{\infty} xpq^{x-1}$$

$$E[X] = p \sum_{x=1}^{\infty} xq^{x-1} = \frac{p}{q} \sum_{x=1}^{\infty} xq^x = \frac{p}{q} \frac{q}{1 - q^2} = \frac{p}{p^2} = \frac{1}{p}.$$

3.5 Averages and Expected Value Families

Let $X \sim \text{Poisson}(\alpha)$, **find** $E[X]$.

$$E[X] = \sum_{x=0}^{\infty} x P_X(x) = \sum_{x=0}^{\infty} x \frac{\alpha^x}{x!} e^{-\alpha}$$

$$x/x! = 1/(x-1)!$$

$$\alpha^x = \alpha \cdot \alpha^{x-1}$$

$$E[X] = \alpha \sum_{x=1}^{\infty} \frac{\alpha^{x-1}}{(x-1)!} e^{-\alpha}.$$

$$l = x - 1$$

$$E[X] = \alpha \underbrace{\sum_{l=0}^{\infty} \frac{\alpha^l}{l!} e^{-\alpha}}_1 = \alpha.$$

$$e^{\alpha} = \sum_{l=0}^{\infty} \alpha^l / l!$$

3.5 Averages and Expected Value Families

(a) For the binomial (n, p) random variable X

$$E[X] = np.$$

(b) For the Pascal (k, p) random variable X

$$E[X] = k/p.$$

(c) For the discrete uniform (k, l) random variable X

$$E[X] = (k + l)/2.$$

3.6 Functions of a Random Variable

Definition

If X is a random variable and $Y = g(X)$, then Y itself is a random variable.

For example: $Y = X^2$.

$$S_X = \{x_1, x_2, x_3, \dots\} \rightarrow S_Y = \{g(x) | x \in S_X\} = \{g(x_1), g(x_2), \dots\}.$$

Then,

$$P_Y(y_k) = P(Y = y_k) = P(g(X) = y_k).$$

3.6 Functions of a Random Variable

Example 3.6.1

Let X be a discrete random variable uniformly distributed with

$$P_X(k) = \frac{1}{5}, \text{ for } k = \{-1, 0, 1, 2, 3\}. \quad \text{Let } Y = 2|X|.$$

a) Find PMF of Y , $P_Y(y)$.

$$S_Y = \{2|x| \text{ where } x \in S_X\} = \{0, 2, 4, 6\}.$$

$$P_Y(0) = P(Y = 0) = P(2|X| = 0) = P(X = 0) = \frac{1}{5}$$

$$\begin{aligned} P_Y(2) &= P(Y = 2) = P(2|X| = 2) = P((X = -1) \text{ or } (X = 1)) \\ &= P_X(-1) + P_X(1) = \frac{1}{5} + \frac{1}{5} = \frac{2}{5} \end{aligned}$$

$$P_Y(4) = P(Y = 4) = P(2|X| = 4) = P(X = 2) + P(X = -2) = \frac{1}{5}$$

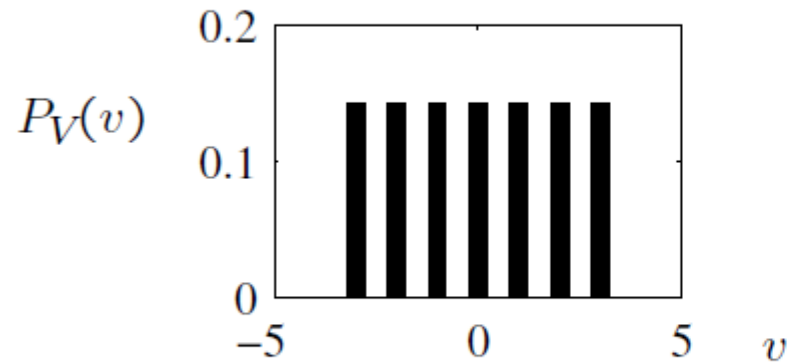
$$P_Y(6) = P(Y = 6) = P(2|X| = 6) = P(X = 3) + P(X = -3) = \frac{1}{5}$$

$$P_Y(k) = \begin{cases} \frac{1}{5} & \text{for } k = 0, 4, 6 \\ \frac{2}{5} & \text{for } k = 2 \\ 0 & \text{otherwise} \end{cases}$$

3.6 Functions of a Random Variable

Example 3.6.2

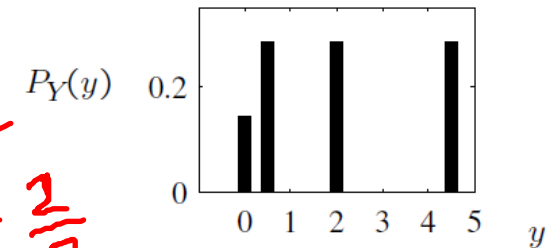
The amplitude V (volts) of a sinusoidal signal is a random variable with PMF



$$P_V(v) = \begin{cases} 1/7 & v = -3, -2, \dots, 3, \\ 0 & \text{otherwise.} \end{cases}$$

Let $Y = V^2/2$ watts denote the power of the transmitted signal. Find $P_Y(y)$.

Handwritten notes: $\{4.5, 2, 0.5, 0, 0.5, 2, 4.5\}$ (possible values of Y)
 $4.5 \rightarrow 2$ (mapping from V=3 to Y=4.5)
 $1/7 + 1/7 = 2/7$ (summing probabilities for Y=0.5, 2, 4.5)
 والباقي نزلنا (the rest we brought down)



The possible values of Y are $S_Y = \{0, 0.5, 2, 4.5\}$

Since $Y = y$ when $V = \sqrt{2y}$ or $V = -\sqrt{2y}$

$$P_Y(y) = P_V(\sqrt{2y}) + P_V(-\sqrt{2y}) = 2/7$$

$$P_Y(0) = P_V(0) = 1/7$$

$$P_Y(y) = \begin{cases} 1/7 & y = 0, \\ 2/7 & y = 0.5, 2, 4.5, \\ 0 & \text{otherwise.} \end{cases}$$

3.6 Functions of a Random Variable

Example 3.6.3

Let X be a discrete random variable with PMF

$$P_X(k) = \begin{cases} 0.2 & \text{for } k = 0 \\ 0.2 & \text{for } k = 1 \\ 0.3 & \text{for } k = 2 \\ 0.3 & \text{for } k = 3 \\ 0 & \text{otherwise} \end{cases}$$

Define $Y = X(X - 1)(X - 2)$. Find the PMF of Y .

$$S_Y = \{x(x - 1)(x - 2) | x \in \{0, 1, 2, 3\}\} = \{0, 6\}$$

$$\begin{aligned} P_Y(0) &= P(Y = 0) = P((X = 0) \text{ or } (X = 1) \text{ or } (X = 2)) \\ &= P_X(0) + P_X(1) + P_X(2) \\ &= 0.7 \end{aligned}$$

$$P_Y(6) = P(X = 3) = 0.3$$

$$P_Y(k) = \begin{cases} 0.7 & \text{for } k = 0 \\ 0.3 & \text{for } k = 6 \\ 0 & \text{otherwise} \end{cases}$$

3.7 Expected Value of a Derived Random Variable

Theorem

Law of the unconscious statistician (LOTUS) for discrete random variables:

$$E[g(X)] = \mu_Y = \sum_{x_k \in S_X} g(x_k) P_X(x_k).$$

Linearity of expectation:

$$Y = aX + b \Rightarrow \underline{E[Y]} = a\underline{E[X]} + b, \quad a, b \in \mathbb{R}$$

3.7 Expected Value of a Derived Random Variable

Example 3.7.1

Let X be a discrete random variable uniformly distributed with

$$P_X(k) = \frac{1}{5}, \text{ for } k = \{-1, 0, 1, 2, 3\}. \quad \text{Let } Y = 2|X|.$$

a) Find PMF of Y , $P_Y(y)$.

b) Find $E[Y]$.

المتغير العشوائي Y يأخذ القيم 0, 2, 4, 6

$$P_Y(k) = \begin{cases} \frac{1}{5} & \text{for } k = 0, 4, 6 \\ \frac{2}{5} & \text{for } k = 2 \\ 0 & \text{otherwise} \end{cases}$$

$$E[X] = -1 * \frac{1}{5} + 0 * \frac{1}{5} + 1 * \frac{1}{5} + 2 * \frac{1}{5} + 3 * \frac{1}{5}$$

$$E[Y] = E[2|X|] = 2|-1| * \frac{1}{5} + 2|0| * \frac{1}{5} + 2|1| * \frac{1}{5} + 2|2| * \frac{1}{5} + 2|3| * \frac{1}{5} = \frac{14}{5}$$

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3.7 Expected Value of a Derived Random Variable

Example 3.7.2

Let $S_X = \{0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi\}$, such that

$$P_X(0) = P_X(\frac{\pi}{4}) = P_X(\frac{\pi}{2}) = P_X(\frac{3\pi}{4}) = P_X(\pi) = \frac{1}{5}$$

Find $E[\sin(X)]$.

$$\begin{aligned} E[g(X)] &= \sum_{x_k \in R_X} g(x_k) P_X(x_k) \\ &= \sin(0) \cdot \frac{1}{5} + \sin(\frac{\pi}{4}) \cdot \frac{1}{5} + \sin(\frac{\pi}{2}) \cdot \frac{1}{5} + \sin(\frac{3\pi}{4}) \cdot \frac{1}{5} + \sin(\pi) \cdot \frac{1}{5} \\ &= 0 \cdot \frac{1}{5} + \frac{\sqrt{2}}{2} \cdot \frac{1}{5} + 1 \cdot \frac{1}{5} + \frac{\sqrt{2}}{2} \cdot \frac{1}{5} + 0 \cdot \frac{1}{5} \\ &= \frac{\sqrt{2}+1}{5}. \end{aligned}$$