

3.8 Variance and Standard Deviation

Definition

The **variance** of a random variable X , with mean $E[X] = \mu_X$, is defined as

$$\text{Var}(X) = E[(X - \mu_X)^2], \quad \mu_X = E[X].$$

The **standard deviation** of a random variable X is defined as

$$\sigma_X = \sqrt{\text{Var}(X)}.$$

3.8 Variance and Standard Deviation

Theorem

Computational formula for the variance:

$$\text{Var}(X) = E[X^2] - (E[X])^2.$$

Proof:

$$\begin{aligned}\text{Var}(X) &= E[(X - \mu_X)^2] \\ &= E[X^2 - 2\mu_X X + \mu_X^2] \\ &= E[X^2] - 2E[\mu_X X] + E[\mu_X^2] \quad \text{by linearity of expectation.} \\ &= E[X^2] - 2\mu_X^2 + \mu_X^2 \\ &= E[X^2] - \mu_X^2.\end{aligned}$$

3.8 Variance and Standard Deviation

Theorem

For a random variable X and real numbers a and b

$$\text{Var}(aX + b) = a^2 \text{Var}(X).$$

Proof:

If $Y = aX + b$, $E[Y] = aE[X] + b$. Thus,

$$\begin{aligned}\text{Var}(Y) &= E[(Y - E[Y])^2] \\ &= E[((aX + b) - (aE[X] + b))^2] \\ &= E[a^2(X - \mu_X)^2] \\ &= a^2 E[(X - \mu_X)^2] = a^2 \text{Var}(X)\end{aligned}$$

3.8 Variance and Standard Deviation

Theorem

If $X_1, X_2, \dots, X_n$ are **independent** random variables and

$X = X_1 + X_2 + \dots + X_n$, then

$$\text{Var}(X) = \text{Var}(X_1) + \text{Var}(X_2) + \dots + \text{Var}(X_n).$$

3.8 Variance and Standard Deviation

Moments

For a random variable X

- The n th moment is $E[X^n]$.
- The n th central moment is $E[(X - \mu_X)^n]$.

3.8 Variance and Standard Deviation

Example 3.8.1

I roll a fair die and let X be the resulting number. Find $E[X]$, $\text{Var}(X)$, and σ_X .

$$S_X = \{1, 2, 3, 4, 5, 6\} \quad P_X(k) = \frac{1}{6} \text{ for } k = 1, 2, \dots, 6.$$

$$E[X] = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6} = \frac{7}{2};$$

$$E[X^2] = 1 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 9 \cdot \frac{1}{6} + 16 \cdot \frac{1}{6} + 25 \cdot \frac{1}{6} + 36 \cdot \frac{1}{6} = \frac{91}{6}.$$

$$\text{Var}(X) = E[X^2] - (E[X])^2 = \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{91}{6} - \frac{49}{4} \approx 2.92,$$

$$\sigma_X = \sqrt{\text{Var}(X)} \approx \sqrt{2.92} \approx 1.71$$

3.8 Variance and Standard Deviation

Example 3.8.2

In an experiment with three customers entering the Phonesmart store, the observation is N , the number of phones purchased. The PMF of N is

$$P_N(n) = \begin{cases} (4 - n)/10 & n = 0, 1, 2, 3 \\ 0 & \text{otherwise.} \end{cases}$$

Find

(a) The expected value $E[N]$

n	0	1	2	3
$P_N(n)$	0.4	0.3	0.2	0.1

$$E[N] = \sum_{n=0}^3 nP_N(n)$$

$$= 0(0.4) + 1(0.3) + 2(0.2) + 3(0.1) = 1.$$

3.8 Variance and Standard Deviation

Example 3.8.2

In an experiment with three customers entering the Phonesmart store, the observation is N , the number of phones purchased. The PMF of N is

$$P_N(n) = \begin{cases} (4 - n)/10 & n = 0, 1, 2, 3 \\ 0 & \text{otherwise.} \end{cases}$$

Find

(b) The second moment $E[N^2]$

$$E[N^2] = \sum_{n=0}^3 n^2 P_N(n) = 0^2(0.4) + 1^2(0.3) + 2^2(0.2) + 3^2(0.1) = 2.$$

(c) The variance $\text{Var}[N]$

$$\text{Var}[N] = E[N^2] - (E[N])^2 = 2 - 1^2 = 1$$

(d) The standard deviation σ_N

$$\sigma_N = \sqrt{\text{Var}[N]} = 1$$

3.8 Variance and Standard Deviation

Example 3.8.3

Let X be a discrete random variable with the following PMF

$$P_X(k) = \begin{cases} 0.1 & \text{for } k = 0 \\ 0.4 & \text{for } k = 1 \\ 0.3 & \text{for } k = 2 \\ 0.2 & \text{for } k = 3 \\ 0 & \text{otherwise} \end{cases}$$

b. Find $\text{Var}(X)$.

a. Find $E[X]$

$$\begin{aligned} E[X] &= \sum_{x_k \in R_X} x_k P_X(x_k) \\ &= 0(0.1) + 1(0.4) + 2(0.3) + 3(0.2) \\ &= 1.6 \end{aligned}$$

c. If $Y = (X - 2)^2$, find EY .

3.8 Variance and Standard Deviation

Example 3.8.3

b. Find $\text{Var}(X)$.

$$\text{Var}(X) = EX^2 - (EX)^2 = EX^2 - (1.6)^2$$

$$EX^2 = 0^2(0.1) + 1^2(0.4) + 2^2(0.3) + 3^2(0.2) = 3.4$$

$$\text{Var}(X) = (3.4) - (1.6)^2 = 0.84$$

c. If $Y = (X - 2)^2$, find EY .

$$E(X - 2)^2 = (0 - 2)^2(0.1) + (1 - 2)^2(0.4) + (2 - 2)^2(0.3) + (3 - 2)^2(0.2) = 1.$$

Selected Problems

3.2.2, 3.2.3, 3.2.5, 3.2.7, **3.2.8**, 3.3.1, 3.3.2, 3.3.3, 3.3.7, 3.3.8, 3.3.10, 3.3.11,
3.3.14, 3.3.15, 3.4.1, 3.4.2, 3.4.3, 3.5.1, 3.5.2, 3.5.3, 3.5.5, 3.6.1, 3.6.2, 3.6.3,
3.6.5, 3.6.6, 3.6.7, 3.7.1, 3.7.3, 3.7.5, 3.7.6, 3.8.1, 3.8.5, 3.8.6, 3.8.7, 3.8.8,
3.8.9.