

EELE 3340

Probability Theory



Chapter 4

Continuous Random Variables

Dr. Mohammed El-Absi

4.1 Definition

Introduction

If $\text{Range}(X)$ is **not** countable then X is a continuous random variable.

Example. $\text{Range}(X) = [a, b]$,

Remember: $[a, b] = \{x \in \mathbb{R}, a \leq x \leq b\}$, $(a, b] = \{x \in \mathbb{R}, a < x \leq b\}$, **etc.**

Example.

- T : Lifetime of a light bulb, $\text{Range}(T) = [0, \infty)$.
- V : Voltage across a resistor, $\text{Range}(V) = [0, V_{max}]$.

Definition. A random variable X having a CDF $F_X(x)$ that is a continuous function for all x in $\mathbb{R}$ is said to be a **continuous random variable**.

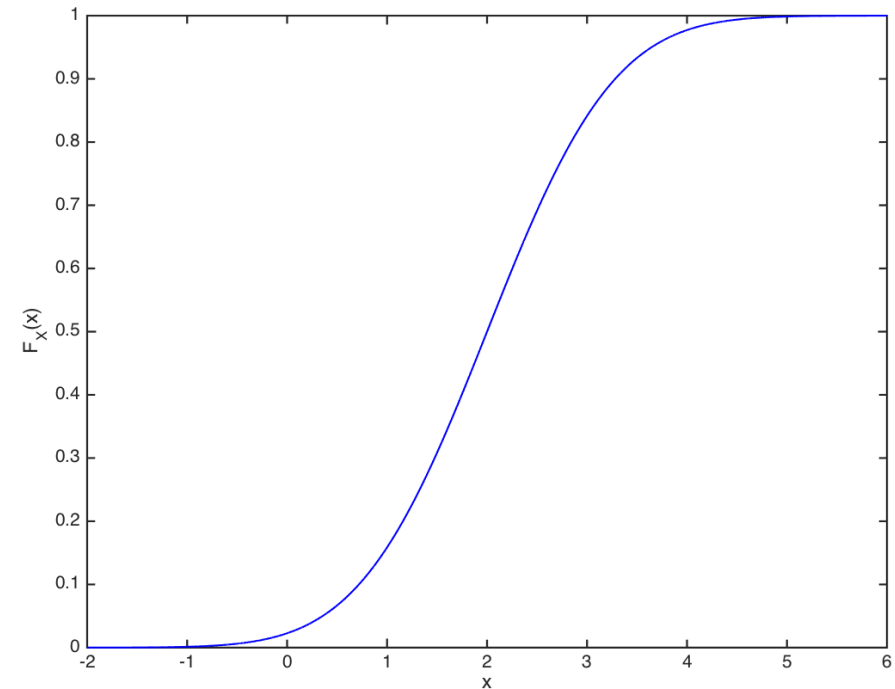
4.2 Cumulative Distributive Function (CDF)

Definition

suppose we have a **continuous** function having those properties – for example:

- A function like this is also a valid CDF
 - . $F_X(x)$ – that is, it can also represent
 - . $P(X \leq x)$ for some random variable X .

$$F_X(x) = \text{Prob}\{X \leq x\}.$$



4.2 Cumulative Distributive Function (CDF)

Example 4.2.1

The cumulative distribution function of the random variable Y is

$$F_Y(y) = \begin{cases} 0 & y < 0, \\ y/4 & 0 \leq y \leq 4, \\ 1 & y > 4. \end{cases}$$

Sketch the CDF of Y and calculate the following probabilities:

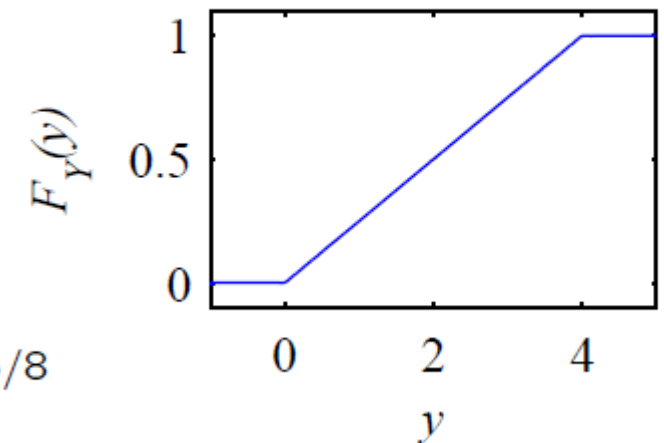
(a) $P[Y \leq -1]$ $F_Y(-1) = 0$

(b) $P[Y \leq 1]$ $F_Y(1) = 1/4$

(c) $P[2 < Y \leq 3] = F_Y(3) - F_Y(2) = 3/4 - 2/4 = 1/4$

(d) $P[Y > 1.5] = 1 - P[Y \leq 1.5] = 1 - F_Y(1.5) = 1 - (1.5)/4 = 5/8$

The CDF of Y is



4.3 Probability Density Function

Introduction

- **For a discrete random variable:** the rate of increase in the CDF $F_X(x)$ is characterized by the PMF $P_X(x)$ - that is, by the locations and sizes of the jumps in the CDF.
- What characterizes the rate of increase for a **continuous function**?

The derivative of the function

4.3 Probability Density Function

Introduction

- If the CDF $F_X(x)$ is a **continuous** function, then X is said to be a **continuous random variable**.
- PMF is Not well-defined for continuous random variables. Instead, we define **probability density function (pdf)**.

4.3 Probability Density Function

Introduction

pdf:

$$f_X(x_1) = \lim_{\Delta \rightarrow 0} \frac{P(x_1 < X \leq x_1 + \Delta)}{\Delta}.$$

Remember that,

$$P(x_1 < X \leq x_1 + \Delta) = F_X(x_1 + \Delta) - F_X(x_1).$$

$$\Rightarrow f_X(x_1) = \lim_{\Delta \rightarrow 0} \frac{F_X(x_1 + \Delta) - F_X(x_1)}{\Delta} = \frac{dF_X(x_1)}{dx_1} = F'_X(x_1).$$

4.3 Probability Density Function

Definition

Consider a continuous random variable X with an absolutely continuous CDF $F_X(x)$. Then we have

$$f_X(x) = \frac{dF_X(x)}{dx} = F'_X(x),$$

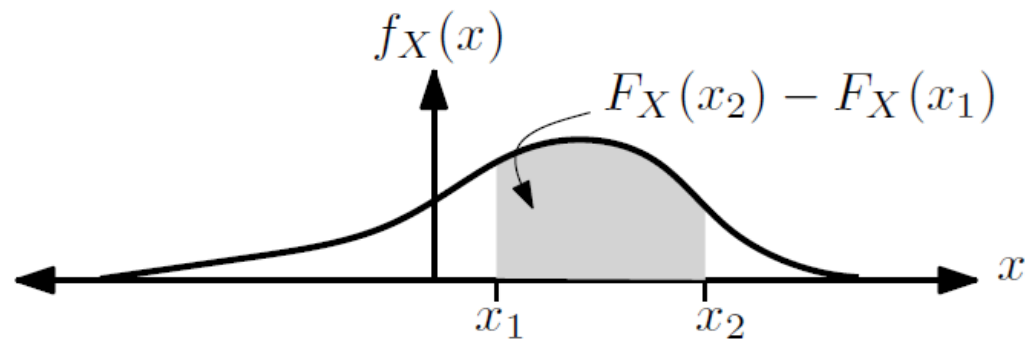
is called the **probability density function (PDF)** of X .

$$F_X(x) = \int_{-\infty}^x f_X(\alpha) d\alpha.$$

4.3 Probability Density Function

Theorem

$$\begin{aligned}P(a < X < b) &= F_X(b) - F_X(a) \\&= \int_{-\infty}^b f_X(\alpha) d\alpha - \int_{-\infty}^a f_X(\alpha) d\alpha \\&= \int_a^b f_X(\alpha) d\alpha.\end{aligned}$$



The PDF and CDF of X .

4.3 Probability Density Function

Comparison with discrete random variable

$$\text{Prob}\{a \leq X < b\} = \sum_{a \leq x_k < b} P_X(x_k) \longrightarrow \int_a^b f_X(x) dx$$

↑discrete ↑continuous

If we integrate over the entire real line, we must get 1, i.e.,

$$\int_{-\infty}^{\infty} f_X(x) dx = 1.$$

4.3 Probability Density Function Properties

1) Since $F_X(x)$ is monotone non-decreasing, its derivative must satisfy

$$f_X(x) \geq 0 \text{ for all } x \in \mathbb{R}.$$

2)
$$\int_{-\infty}^{\infty} f_X(u) du = F_X(\infty) - F_X(-\infty) = 1 - 0 = 1.$$

4.3 Probability Density Function

Example 4.3.2

Random variable X has probability density function

$$f_X(x) = \begin{cases} cxe^{-x/2} & x \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

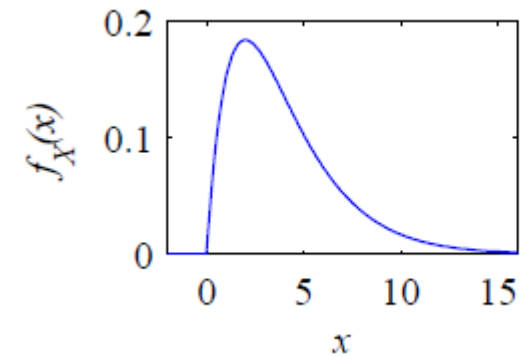
Sketch the PDF and find the following:

(a) the constant c

$$1 = \int_{-\infty}^{\infty} f_X(x) dx = \int_0^{\infty} cxe^{-x/2} dx.$$

$$1 = \underbrace{-2cxe^{-x/2}}_{=0} \Big|_0^{\infty} + \int_0^{\infty} 2ce^{-x/2} dx = -4ce^{-x/2} \Big|_0^{\infty} = 4c.$$
$$c = 1/4$$

$$f_X(x) = \begin{cases} (x/4)e^{-x/2} & x \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$



$$\int u v dx = u \int v dx - \int u' (\int v dx) dx$$

- u is the function $u(x)$
- v is the function $v(x)$
- u' is the derivative of the function $u(x)$

4.3 Probability Density Function

Example 4.3.2

(b) the CDF $F_X(x)$

$$F_X(x) = 0 \text{ for all } x < 0$$

$$f_X(x) = \begin{cases} (x/4)e^{-x/2} & x \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

$$\begin{aligned} F_X(x) &= \int_0^x f_X(y) dy = \int_0^x \frac{y}{4} e^{-y/2} dy \\ &= -\frac{y}{2} e^{-y/2} \Big|_0^x + \int_0^x \frac{1}{2} e^{-y/2} dy = 1 - \frac{x}{2} e^{-x/2} - e^{-x/2} \end{aligned}$$

$$F_X(x) = \begin{cases} 1 - (\frac{x}{2} + 1) e^{-x/2} & x \geq 0, \\ 0 & \text{ow.} \end{cases}$$

4.3 Probability Density Function

Example 4.3.2

$$(c) \ P[0 \leq X \leq 4]$$

$$P[0 \leq X \leq 4] = F_X(4) - F_X(0) = 1 - 3e^{-2}$$

$$(d) \ P[-2 \leq X \leq 2]$$

$$P[-2 \leq X \leq 2] = F_X(2) - F_X(-2)$$

$$= 1 - 3e^{-1}$$

4.4 Expected Values

Definitions

$$\sum_{k=-\infty}^{\infty} \longrightarrow \int_{-\infty}^{\infty}; \quad P_X(x_k) \longrightarrow f_X(x),$$

So, the expected value of a continuous random variable X is

$$E[X] = \int_{-\infty}^{\infty} x f_X(x) dx.$$

Law of the unconscious statistician (LOTUS) for continuous random variables:

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) dx.$$

4.4 Expected Values

Definitions

Remember that the variance of any random variable is defined as

$$\text{Var}(X) = E[(X - \mu_X)^2] = E[X^2] - (E[X])^2.$$

So for a continuous random variable, we can write

$$\begin{aligned}\text{Var}(X) &= E[(X - \mu_X)^2] = \int_{-\infty}^{\infty} (x - \mu_X)^2 f_X(x) dx \\ &= E[X^2] - (E[X])^2 = \int_{-\infty}^{\infty} x^2 f_X(x) dx - \mu_X^2.\end{aligned}$$

4.4 Expected Values

Example 4.4.1

The probability density function of the random variable Y is

$$f_Y(y) = \begin{cases} 3y^2/2 & -1 \leq y \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

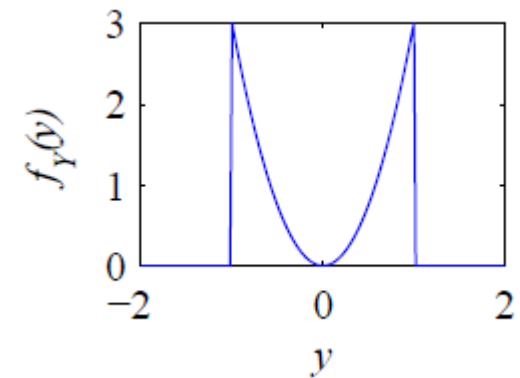
Sketch the PDF and find the following:

(a) the expected value $E[Y]$

$$E[Y] = \int_{-\infty}^{\infty} y f_Y(y) dy = \int_{-1}^1 (3/2)y^3 dy = (3/8)y^4 \Big|_{-1}^1 = 0.$$

(b) the second moment $E[Y^2]$

$$E[Y^2] = \int_{-\infty}^{\infty} y^2 f_Y(y) dy = \int_{-1}^1 (3/2)y^4 dy = (3/10)y^5 \Big|_{-1}^1 = 3/5.$$



4.4 Expected Values

Example 4.4.1

(c) the variance $\text{Var}[Y]$

$$\text{Var}[Y] = E[Y^2] - (E[Y])^2 = 3/5$$

(d) the standard deviation σ_Y

$$\sigma_Y = \sqrt{\text{Var}[Y]} = \sqrt{3/5}.$$