

4.6 Gaussian Random Variables

Definition

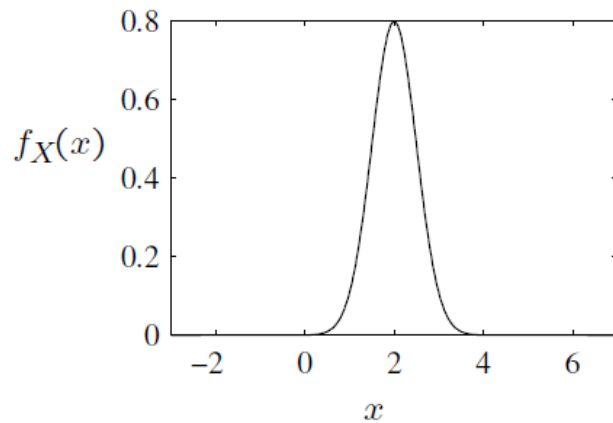
Gaussian function

$$f(x) = a \cdot \exp\left(-\frac{(x-b)^2}{2c^2}\right)$$

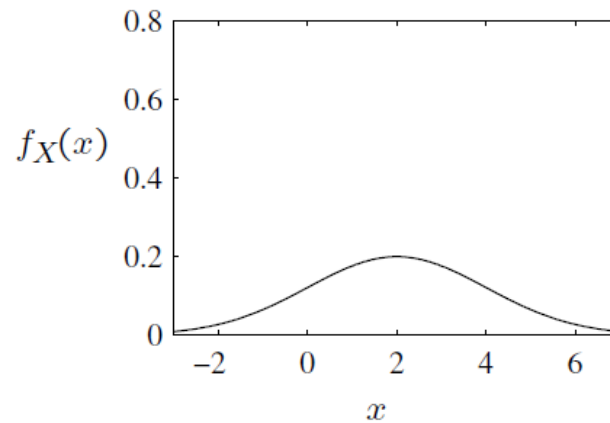
X is a Gaussian (μ, σ) random variable if the PDF of X is

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2},$$

where the parameter μ can be any real number and the parameter $\sigma > 0$.



(a) $\mu = 2, \sigma = 1/2$



(b) $\mu = 2, \sigma = 2$

Two examples of a Gaussian random variable X with expected value μ and standard deviation σ .

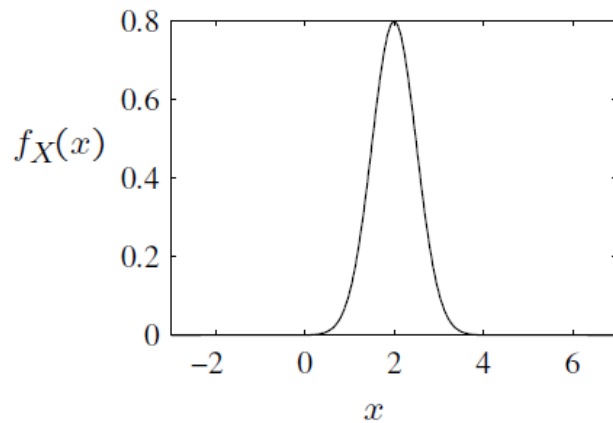
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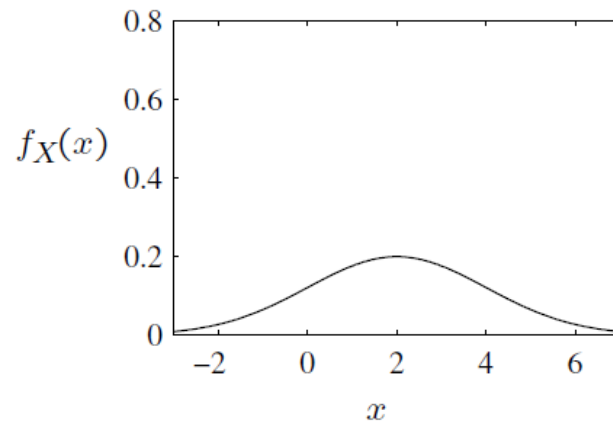
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4.6 Gaussian Random Variables

Theorem

If $X \sim \text{Gaussian}(\mu_X, \sigma_X)$, and $Y = aX + b$, where $a, b \in \mathbb{R}$, then $Y \sim \text{Gaussian}(\mu_Y, \sigma_Y)$, where

$$\mu_Y = a\mu_X + b, \quad \sigma_Y^2 = a^2\sigma_X^2.$$

$$E[Y] = E[aX + b] = aEX + b = a\mu + b,$$

$$\text{Var}(Y) = \text{Var}(aX + b) = a^2\text{Var}(X) = a^2\sigma^2.$$

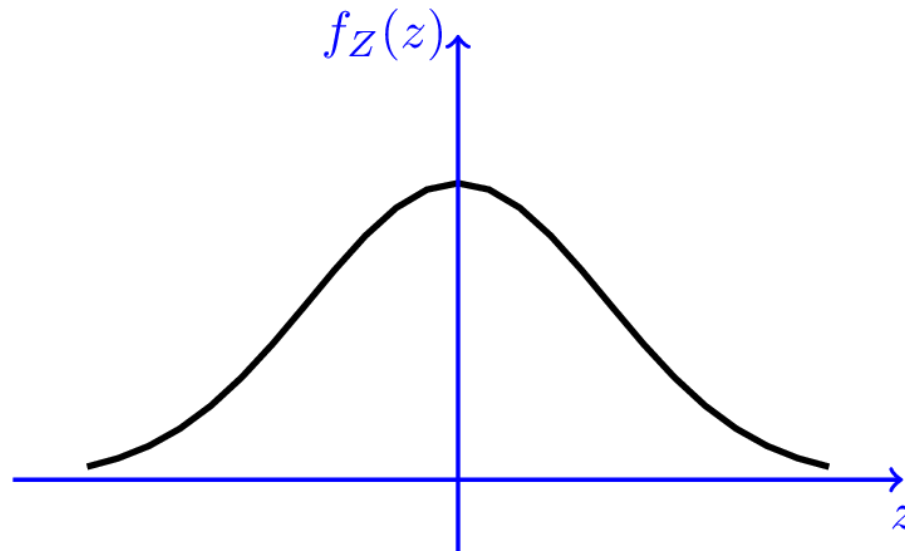
4.6 Gaussian Random Variables

Definition: Standard Normal Random

The standard normal random variable Z is the Gaussian $(0, 1)$ random variable.

$$Z \sim N(0, 1)$$

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}, \quad \text{for all } z \in \mathbb{R}.$$

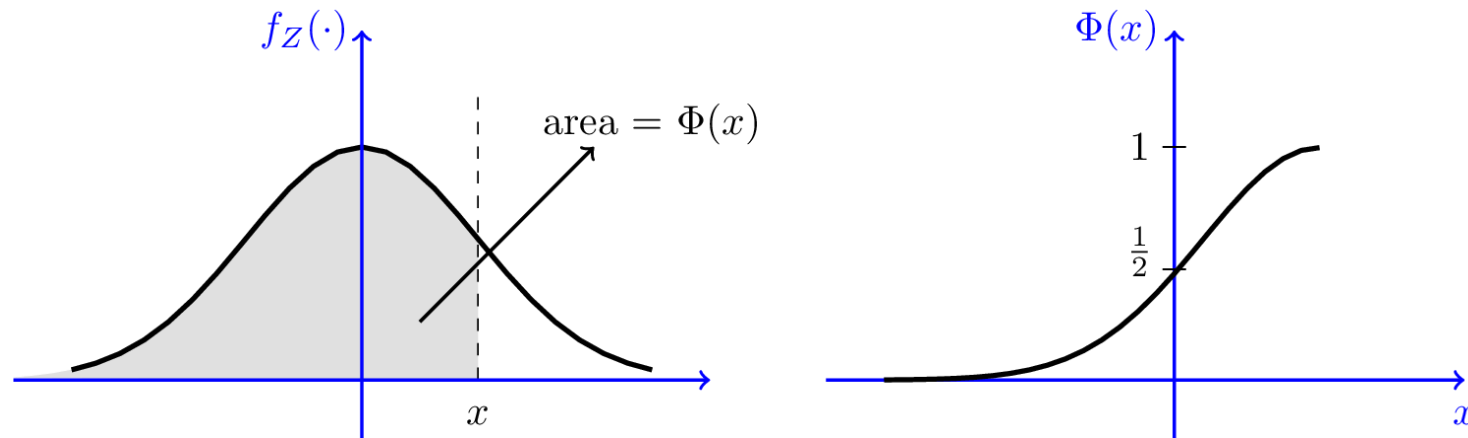


4.6 Gaussian Random Variables

Standard Normal Random

CDF of the standard normal: The CDF of the standard normal distribution is denoted by the Φ function.

$$\Phi(x) = P(X \leq x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{u^2}{2}} du.$$



4.6 Gaussian Random Variables

Standard Normal Random

If X is a Gaussian (μ, σ) random variable, the CDF of X is

$$F_X(x) = \Phi\left(\frac{x - \mu}{\sigma}\right).$$

The probability that X is in the interval $(a, b]$ is

$$\mathbb{P}[a < X \leq b] = \Phi\left(\frac{b - \mu}{\sigma}\right) - \Phi\left(\frac{a - \mu}{\sigma}\right).$$

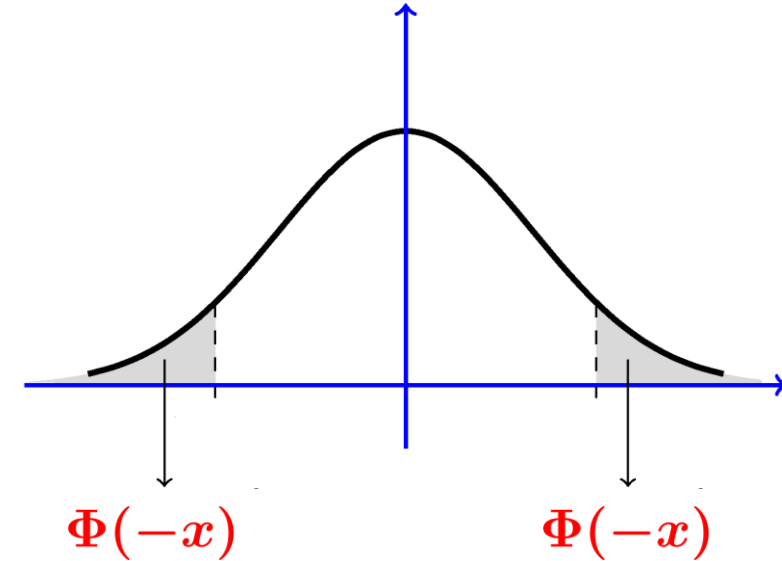
z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$
0.00	0.5000	0.50	0.6915	1.00	0.8413	1.50	0.9332	2.00	0.97725	2.50	0.99379
0.01	0.5040	0.51	0.6950	1.01	0.8438	1.51	0.9346	2.01	0.97778	2.51	0.99396
0.02	0.5080	0.52	0.6985	1.02	0.8461	1.52	0.9357	2.02	0.97831	2.52	0.99413
0.03	0.5120	0.53	0.7019	1.03	0.8485	1.53	0.9370	2.03	0.97882	2.53	0.99430
0.04	0.5160	0.54	0.7054	1.04	0.8508	1.54	0.9382	2.04	0.97932	2.54	0.99446
0.05	0.5199	0.55	0.7088	1.05	0.8531	1.55	0.9394	2.05	0.97982	2.55	0.99461
0.06	0.5239	0.56	0.7123	1.06	0.8554	1.56	0.9406	2.06	0.98030	2.56	0.99477
0.07	0.5279	0.57	0.7157	1.07	0.8577	1.57	0.9418	2.07	0.98077	2.57	0.99492
0.08	0.5319	0.58	0.7190	1.08	0.8599	1.58	0.9429	2.08	0.98124	2.58	0.99506
0.09	0.5359	0.59	0.7224	1.09	0.8621	1.59	0.9441	2.09	0.98169	2.59	0.99520
0.10	0.5398	0.60	0.7257	1.10	0.8643	1.60	0.9452	2.10	0.98214	2.60	0.99534
0.11	0.5438	0.61	0.7291	1.11	0.8665	1.61	0.9463	2.11	0.98257	2.61	0.99547
0.12	0.5478	0.62	0.7324	1.12	0.8686	1.62	0.9474	2.12	0.98300	2.62	0.99560
0.13	0.5517	0.63	0.7357	1.13	0.8708	1.63	0.9484	2.13	0.98341	2.63	0.99573
0.14	0.5557	0.64	0.7389	1.14	0.8729	1.64	0.9495	2.14	0.98382	2.64	0.99585
0.15	0.5596	0.65	0.7422	1.15	0.8749	1.65	0.9505	2.15	0.98422	2.65	0.99598
0.16	0.5636	0.66	0.7454	1.16	0.8770	1.66	0.9515	2.16	0.98461	2.66	0.99609
0.17	0.5675	0.67	0.7486	1.17	0.8790	1.67	0.9525	2.17	0.98500	2.67	0.99621
0.18	0.5714	0.68	0.7517	1.18	0.8810	1.68	0.9535	2.18	0.98537	2.68	0.99632
0.19	0.5753	0.69	0.7549	1.19	0.8830	1.69	0.9545	2.19	0.98574	2.69	0.99643
0.20	0.5793	0.70	0.7580	1.20	0.8849	1.70	0.9554	2.20	0.98610	2.70	0.99653
0.21	0.5832	0.71	0.7611	1.21	0.8869	1.71	0.9564	2.21	0.98645	2.71	0.99664
0.22	0.5871	0.72	0.7642	1.22	0.8888	1.72	0.9573	2.22	0.98679	2.72	0.99674
0.23	0.5910	0.73	0.7673	1.23	0.8907	1.73	0.9582	2.23	0.98713	2.73	0.99683
0.24	0.5948	0.74	0.7704	1.24	0.8925	1.74	0.9591	2.24	0.98745	2.74	0.99693
0.25	0.5987	0.75	0.7734	1.25	0.8944	1.75	0.9599	2.25	0.98778	2.75	0.99702
0.26	0.6026	0.76	0.7764	1.26	0.8962	1.76	0.9608	2.26	0.98809	2.76	0.99711
0.27	0.6064	0.77	0.7794	1.27	0.8980	1.77	0.9616	2.27	0.98840	2.77	0.99720
0.28	0.6103	0.78	0.7823	1.28	0.8997	1.78	0.9625	2.28	0.98870	2.78	0.99728
0.29	0.6141	0.79	0.7852	1.29	0.9015	1.79	0.9633	2.29	0.98899	2.79	0.99736
0.30	0.6179	0.80	0.7881	1.30	0.9032	1.80	0.9641	2.30	0.98928	2.80	0.99744
0.31	0.6217	0.81	0.7910	1.31	0.9049	1.81	0.9649	2.31	0.98956	2.81	0.99752
0.32	0.6255	0.82	0.7939	1.32	0.9066	1.82	0.9656	2.32	0.98983	2.82	0.99760
0.33	0.6293	0.83	0.7967	1.33	0.9082	1.83	0.9664	2.33	0.99010	2.83	0.99767
0.34	0.6331	0.84	0.7995	1.34	0.9099	1.84	0.9671	2.34	0.99036	2.84	0.99774
0.35	0.6368	0.85	0.8023	1.35	0.9115	1.85	0.9678	2.35	0.99061	2.85	0.99781
0.36	0.6406	0.86	0.8051	1.36	0.9131	1.86	0.9686	2.36	0.99086	2.86	0.99788
0.37	0.6443	0.87	0.8078	1.37	0.9147	1.87	0.9693	2.37	0.99111	2.87	0.99795
0.38	0.6480	0.88	0.8106	1.38	0.9162	1.88	0.9699	2.38	0.99134	2.88	0.99801
0.39	0.6517	0.89	0.8133	1.39	0.9177	1.89	0.9706	2.39	0.99158	2.89	0.99807
0.40	0.6554	0.90	0.8159	1.40	0.9192	1.90	0.9713	2.40	0.99180	2.90	0.99813
0.41	0.6591	0.91	0.8186	1.41	0.9207	1.91	0.9719	2.41	0.99202	2.91	0.99819
0.42	0.6628	0.92	0.8212	1.42	0.9222	1.92	0.9726	2.42	0.99224	2.92	0.99825
0.43	0.6664	0.93	0.8238	1.43	0.9236	1.93	0.9732	2.43	0.99245	2.93	0.99831
0.44	0.6700	0.94	0.8264	1.44	0.9251	1.94	0.9738	2.44	0.99266	2.94	0.99836
0.45	0.6736	0.95	0.8289	1.45	0.9265	1.95	0.9744	2.45	0.99286	2.95	0.99841
0.46	0.6772	0.96	0.8315	1.46	0.9279	1.96	0.9750	2.46	0.99305	2.96	0.99846
0.47	0.6808	0.97	0.8340	1.47	0.9292	1.97	0.9756	2.47	0.99324	2.97	0.99851
0.48	0.6844	0.98	0.8365	1.48	0.9306	1.98	0.9761	2.48	0.99343	2.98	0.99856
0.49	0.6879	0.99	0.8389	1.49	0.9319	1.99	0.9767	2.49	0.99361	2.99	0.99861

Table 4.2 The standard normal CDF $\Phi(y)$.

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properties of the Φ function

- 1) $\lim_{x \rightarrow \infty} \Phi(x) = 1, \quad \lim_{x \rightarrow -\infty} \Phi(x) = 0;$
- 2) $\Phi(0) = \frac{1}{2};$
- 3) $\Phi(-x) = 1 - \Phi(x),$ for all $x \in \mathbb{R};$
- 4) $\Phi(-x) + \Phi(x) = 1.$

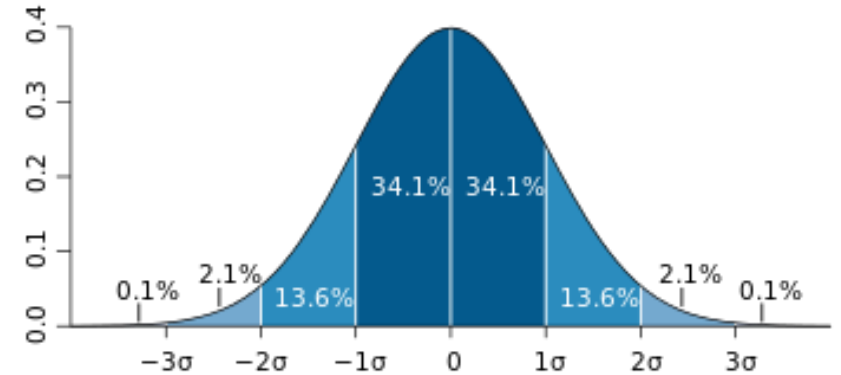


4.6 Gaussian Random Variables

Properties

Suppose that $X \sim N(\mu, \sigma)$. Then:

$$\begin{aligned} P(|X - \mu| \leq \sigma) &= P(\mu - \sigma \leq X \leq \mu + \sigma) \\ &= \Phi\left(\frac{\mu + \sigma - \mu}{\sigma}\right) - \Phi\left(\frac{\mu - \sigma - \mu}{\sigma}\right) \\ &= \Phi(1) - \Phi(-1) = 2\Phi(1) - 1 = 0.6827 \end{aligned}$$



So, the probability that a Gaussian random variable takes a value within 1 standard deviation from its mean is 0.6827.

$$\begin{aligned} P(|X - \mu| \leq 3\sigma) &= P(\mu - 3\sigma \leq X \leq \mu + 3\sigma) \\ &= \Phi\left(\frac{\mu + 3\sigma - \mu}{\sigma}\right) - \Phi\left(\frac{\mu - 3\sigma - \mu}{\sigma}\right) \\ &= \Phi(3) - \Phi(-3) = 2\Phi(3) - 1 = 0.9973 \end{aligned}$$

The probability that a Gaussian random variable takes a value within 3 standard deviations from its mean is 0.9973.

4.6 Gaussian Random Variables

Complementary CDF

The standard normal complementary CDF is

$$Q(z) = \text{P}[Z > z] = \frac{1}{\sqrt{2\pi}} \int_z^{\infty} e^{-u^2/2} du = 1 - \Phi(z).$$

4.6 Gaussian Random Variables

Example 4.6.1

Let $X \sim N(-5, 2)$

a. Find $P(X < 0)$.

$$P(X < 0) = F_X(0) = \Phi\left(\frac{0 - (-5)}{2}\right) = \Phi(2.5) \approx 0.99$$

b. Find $P(-7 < X < -3)$:

$$\begin{aligned} P(-7 < X < -3) &= F_X(-3) - F_X(-7) \\ &= \Phi\left(\frac{(-3) - (-5)}{2}\right) - \Phi\left(\frac{(-7) - (-5)}{2}\right) \\ &= \Phi(1) - \Phi(-1) = 2\Phi(1) - 1 \quad (\text{since } \Phi(-x) = 1 - \Phi(x)) \\ &\approx 0.68 \end{aligned}$$

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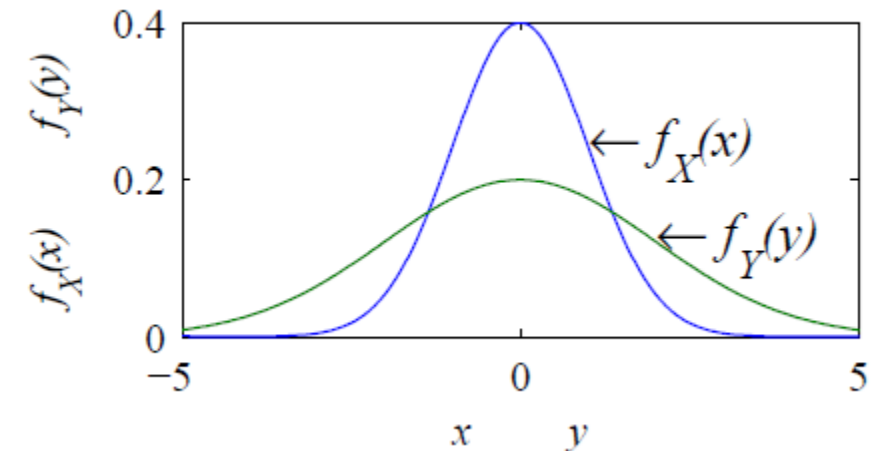
Example 4.6.2

X is the Gaussian $(0,1)$ random variable and Y is the Gaussian $(0,2)$ random variable. Sketch the PDFs $f_X(x)$ and $f_Y(y)$ on the same axes and find:

(a) $P[-1 < X \leq 1]$,

$$\begin{aligned} P[-1 < X \leq 1] &= F_X(1) - F_X(-1) \\ &= \Phi(1) - \Phi(-1) \\ &= 2\Phi(1) - 1 \\ &= 0.6826 \end{aligned}$$

The PDFs of X and Y are:



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Example 4.6.2

$$(b) \ P[-1 < Y \leq 1]$$

$$\begin{aligned} P[-1 < Y \leq 1] &= F_Y(1) - F_Y(-1) \\ &= \Phi\left(\frac{1}{\sigma_Y}\right) - \Phi\left(\frac{-1}{\sigma_Y}\right) \\ &= 2\Phi\left(\frac{1}{2}\right) - 1 = 0.383 \end{aligned}$$

$$(c) \ P[X > 3.5] \quad P[X > 3.5] = Q(3.5) = 2.33 \times 10^{-4}$$

$$(d) \ P[Y > 3.5]$$

$$P[Y > 3.5] = Q\left(\frac{3.5}{2}\right) = 1 - \Phi(1.75) = 0.04$$

Selected Problems

4.2.1, 4.2.2, 4.2.4, 4.3.1, 4.3.2, 4.3.5, 4.3.6, 4.4.1 - 4.4.8, 4.5.1 – 4.5.13,
4.6.1 – 4.6.7, 6.6.9, 4.6.11,