

EELE 3340

Probability Theory



Chapter 5

Pairs of Random Variables

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5.1 Joint Probability Mass Function

Introduction

We would like to measure random variable X , but we instead observe

$$Y = X + Z$$

The noise Z prevents us from perfectly observing X . In some settings, Z is an interfering signal. In the simplest setting, Z is just noise inside the circuitry of your measurement device that is unrelated to X . In this case, it is appropriate to assume that the signal and noise are independent; that is, the events $X = x$ and $Z = z$ are independent. This simple model produces three random variables, X , Y and Z , but any pair completely specifies the remaining random variable. Thus we will see that a probability model for the pair (X, Z) or for the pair (X, Y) will be sufficient to analyze experiments related to this system.

5.1-3 Joint Probability Mass Function

Definition

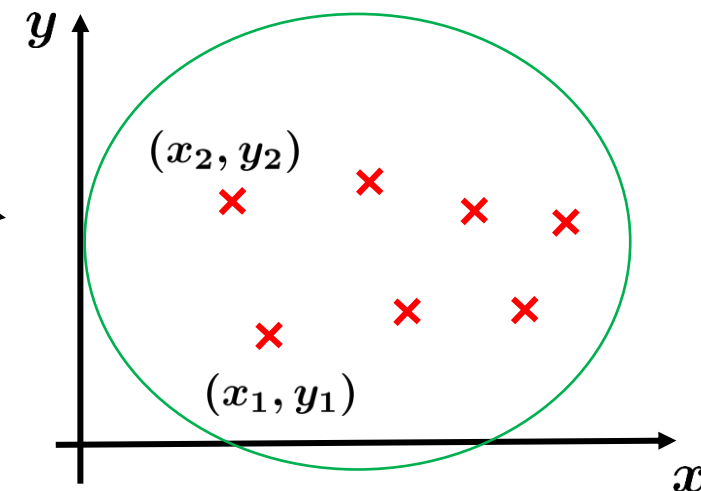
Joint Probability Mass Function (PMF) for X and Y :

$$P_{XY}(x_j, y_j) = P(X = x_k, Y = y_j).$$

R_{XY} = all possible value for (X, Y) .

$$= \{(x_i, y_j) | x_i \in R_X, y_j \in R_Y\}.$$

$$\sum_{(x_i, y_j) \in R_{XY}} P_{XY}(x_i, y_j) = 1$$



5.1-3 Joint Probability Mass Function

Example 1

Consider two random variables X and Y with joint PMF given

a. Find $P(X = 0, Y \leq 1)$.

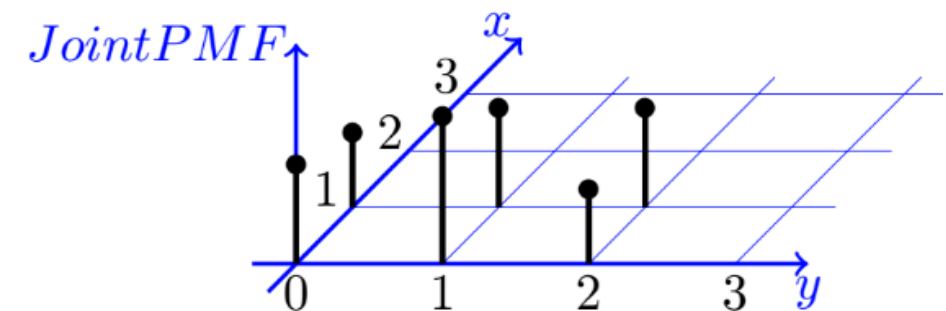
$$P(X = 0, Y \leq 1) = P_{XY}(0, 0) + P_{XY}(0, 1) = \frac{1}{6} + \frac{1}{4} = \frac{5}{12}$$

b. Find the marginal PMFs of X and Y .

$$R_X = \{0, 1\} \quad \text{and} \quad R_Y = \{0, 1, 2\}.$$

$$P_X(0) = P_{XY}(0, 0) + P_{XY}(0, 1) + P_{XY}(0, 2) = \frac{1}{6} + \frac{1}{4} + \frac{1}{8} = \frac{13}{24}.$$

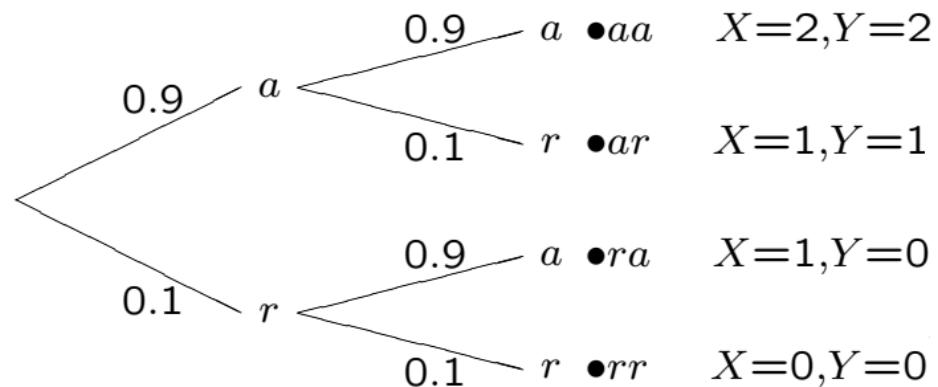
	$Y = 0$	$Y = 1$	$Y = 2$
$X = 0$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{8}$
$X = 1$	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{6}$



5.1-3 Joint Probability Mass Function

Example 2

Test two integrated circuits one after the other. On each test, the possible outcomes are a (accept) and r (reject). Assume that all circuits are acceptable with probability 0.9 and that the outcomes of successive tests are independent. Count the number of acceptable circuits X and count the number of successful tests Y before you observe the first reject. (If both tests are successful, let $Y = 2$.) Draw a tree diagram for the experiment and find the joint PMF $P_{X,Y}(x,y)$.



5.1-3 Joint Probability Mass Function

Example 2

$$S = \{aa, ar, ra, rr\}$$

$$P[aa] = 0.81,$$

$$P[ra] = 0.09,$$

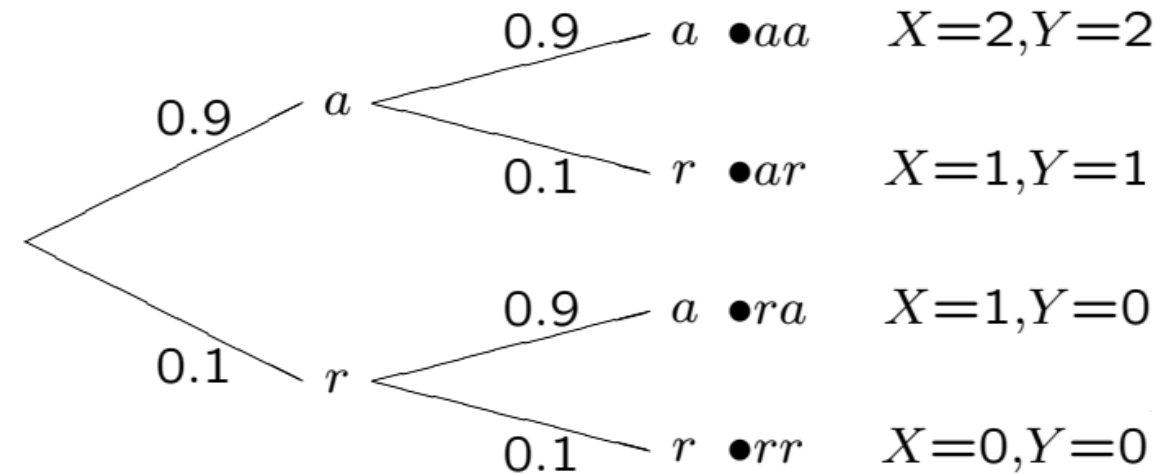
$$P[ar] = 0.09,$$

$$P[rr] = 0.01.$$

$$g(aa) = (2, 2) \quad g(ar) = (1, 1)$$

$$g(ra) = (1, 0) \quad g(rr) = (0, 0)$$

$P_{X,Y}(x, y)$	$y = 0$	$y = 1$	$y = 2$
$x = 0$	0.01	0	0
$x = 1$	0.09	0.09	0
$x = 2$	0	0	0.81



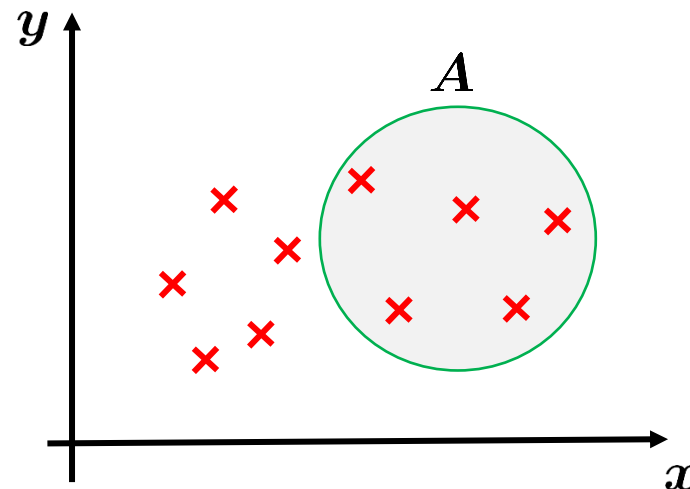
$$P_{X,Y}(x, y) = \begin{cases} 0.81 & x = 2, y = 2, \\ 0.09 & x = 1, y = 1, \\ 0.09 & x = 1, y = 0, \\ 0.01 & x = 0, y = 0. \\ 0 & \text{otherwise} \end{cases}$$

5.1-3 Joint Probability Mass Function

Theorem

For discrete random variables X and Y and any set B in the X, Y plane,
the probability of the event $\{(X, Y) \in B\}$ is

$$P[B] = \sum_{(x,y) \in B} P_{X,Y}(x,y).$$



5.1-3 Joint Probability Mass Function

Example 3

The joint PMF $P_{Q,G}(q, g)$ for random variables Q and G is given in the following table:

$P_{Q,G}(q, g)$	$g = 0$	$g = 1$	$g = 2$	$g = 3$
$q = 0$	0.06	0.18	0.24	0.12
$q = 1$	0.04	0.12	0.16	0.08

Calculate the following probabilities:

(a) $P[Q = 0]$

$$\begin{aligned} P[Q = 0] &= P_{Q,G}(0, 0) + P_{Q,G}(0, 1) + P_{Q,G}(0, 2) + P_{Q,G}(0, 3) \\ &= 0.06 + 0.18 + 0.24 + 0.12 = 0.6 \end{aligned}$$

5.1-3 Joint Probability Mass Function

Example 3

(b) $P[Q = G]$

$P_{Q,G}(q, g)$	$g = 0$	$g = 1$	$g = 2$	$g = 3$
$q = 0$	0.06	0.18	0.24	0.12
$q = 1$	0.04	0.12	0.16	0.08

$$P[Q = G] = P_{Q,G}(0, 0) + P_{Q,G}(1, 1) = 0.18$$

(c) $P[G > 1]$

$$\begin{aligned} P[G > 1] &= \sum_{g=2}^3 \sum_{q=0}^1 P_{Q,G}(q, g) \\ &= \underline{0.24} + \underline{0.16} + \underline{0.12} + \underline{0.08} = 0.6 \end{aligned}$$

(d) $P[G > Q]$

$$\begin{aligned} P[G > Q] &= \sum_{q=0}^1 \sum_{g=q+1}^3 P_{Q,G}(q, g) \\ &= 0.18 + 0.24 + 0.12 + 0.16 + 0.08 = 0.78 \end{aligned}$$

5.1-3 Marginal PMF

Definition

For discrete random variables X and Y with joint PMF $P_{X,Y}(x,y)$,

$$\underline{P_X(x)} = \sum_{\underline{y \in S_Y}} P_{X,Y}(x,y), \quad \underline{P_Y(y)} = \sum_{\underline{x \in S_X}} P_{X,Y}(x,y).$$

5.1-3 Marginal PMF

Example 4

Back to Example 2, Find the marginal PMFs for the random variables X and Y .

$P_{X,Y}(x, y)$	$y = 0$	$y = 1$	$y = 2$
$x = 0$	0.01	0	0
$x = 1$	0.09	0.09	0
$x = 2$	0	0	0.81

We note that both X and Y have range $\{0, 1, 2\}$.

$$P_X(0) = \sum_{y=0}^2 P_{X,Y}(0, y) = 0.01$$

$$P_X(1) = \sum_{y=0}^2 P_{X,Y}(1, y) = 0.18$$

$$P_X(2) = \sum_{y=0}^2 P_{X,Y}(2, y) = 0.81$$

$$P_X(x) = 0 \quad x \neq 0, 1, 2$$

otherwise

$P_{X,Y}(x, y)$	$y = 0$	$y = 1$	$y = 2$	$P_X(x)$
$x = 0$	0.01	0	0	0.01
$x = 1$	0.09	0.09	0	0.18
$x = 2$	0	0	0.81	0.81
$P_Y(y)$	0.10	0.09	0.81	

5.1-3 Joint Cumulative Distribution Function

Definition

Remember that, for a random variable X , we define the CDF as

$$F_X(x) = P(X \leq x).$$

The **joint cumulative distribution function** of two random variables

$$F_{XY}(x, y) = P(X \leq x, Y \leq y).$$



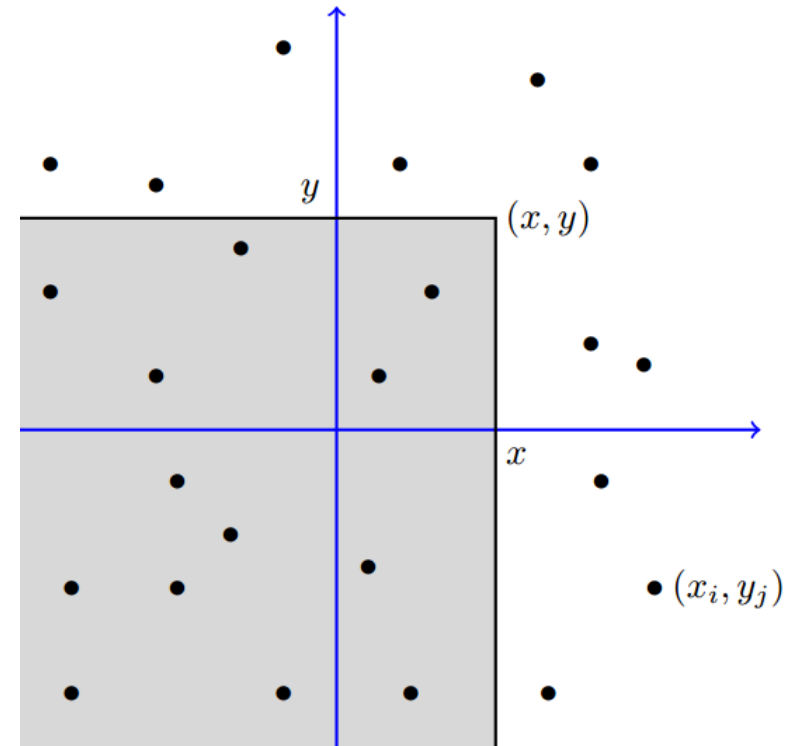
5.1-3 Joint Cumulative Distribution Function

Definition

$$F_{XY}(x, y) = P(\text{shaded region}).$$



$$F_{XY}(1, 2) = P(X \leq 1, Y \leq 2).$$



5.1-3 Joint Cumulative Distribution Function

Properties

خاصة الـ Joint CDF

$$\underline{F_{XY}(x, \infty)} = P(X \leq x, Y < \infty) = \underline{P(X \leq x)} = \underline{F_X(x)},$$

$$F_{XY}(\infty, y) = P(X < \infty, Y < y) = \underline{P(Y \leq y)} = \underline{F_Y(y)},$$

$$\underline{F_{XY}(\infty, \infty)} = 1,$$

$$\underline{F_{XY}(-\infty, y)} = 0, \quad \text{for any } \underline{y},$$

$$\underline{F_{XY}(x, -\infty)} = 0, \quad \text{for any } \underline{x}.$$

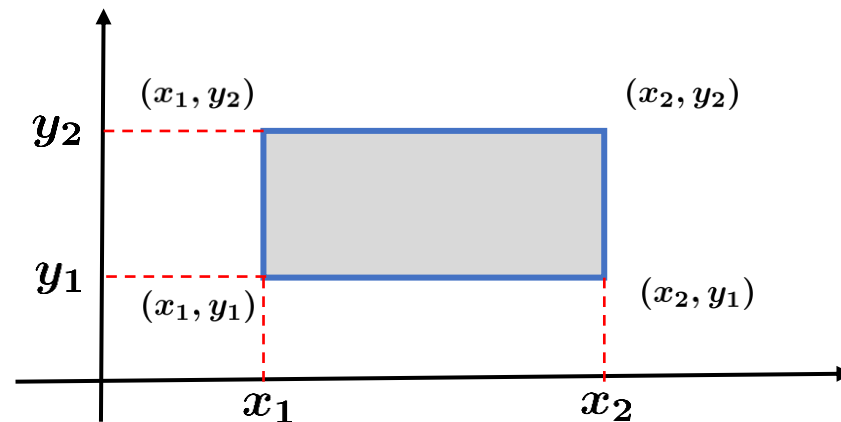
$$0 \leq \underline{F_{XY}(x, y)} \leq 1$$

5.1-3 Joint Cumulative Distribution Function

Lemma

For two random variables X and Y , and real numbers $x_1 \leq x_2$, $y_1 \leq y_2$, we have

$$P(x_1 < X \leq x_2, y_1 < Y \leq y_2) = F_{XY}(x_2, y_2) - F_{XY}(x_1, y_2) - F_{XY}(x_2, y_1) + F_{XY}(x_1, y_1).$$



5.1-3 Joint Cumulative Distribution Function

Example 5

X years is the age of children entering first grade in a school. Y years is the age of children entering second grade. The joint CDF of X and Y is

$$F_{X,Y}(x,y) = \begin{cases} 0 & x < 5, \\ 0 & y < 6, \\ (x-5)(y-6) & 5 \leq x < 6, 6 \leq y < 7, \\ y-6 & x \geq 6, 6 \leq y < 7, \\ x-5 & 5 \leq x < 6, y \geq 7, \\ 1 & \text{otherwise.} \end{cases}$$

Find $F_X(x)$ and $F_Y(y)$.

$$F_X(x) = \begin{cases} 0 & x < 5, \\ x-5 & 5 \leq x < 6, \\ 1 & x \geq 6, \end{cases} \quad F_Y(y) = \begin{cases} 0 & y < 6, \\ y-6 & 6 \leq y < 7, \\ 1 & y \geq 7. \end{cases}$$