

5.4 Joint Probability Density Function

Definition

Two random variables X and Y are **jointly continuous** if there exists a nonnegative function $f_{XY} : \mathbb{R}^2 \rightarrow \mathbb{R}$, such that, for any set $A \in \mathbb{R}^2$, we have

$$P((X, Y) \in A) = \iint_A f_{XY}(x, y) dx dy,$$

The function $f_{XY}(x, y)$ is called **the joint probability density function (PDF)** of X and Y .

5.4 Joint Probability Density Function

Definition

CDF:
$$F_X(x) = P(X \leq x) = \int_{-\infty}^x f_X(u) du,$$

Joint CDF:
$$F_{XY}(x, y) = P(X \leq x, Y \leq y) = \int_{-\infty}^y \int_{-\infty}^x f_{XY}(u, v) du dv,$$

and

$$f_X(x) = \frac{d}{dx} F_X(x),$$

$$f_{XY}(x, y) = \frac{\partial^2}{\partial x \partial y} F_{XY}(x, y).$$

5.4 Joint Probability Density Function

Properties

The **joint cumulative function** of two random variables X and Y is defined as

$$F_{XY}(x, y) = P(X \leq x, Y \leq y).$$

The joint CDF satisfies the following properties:

- 1) $F_X(x) = F_{XY}(x, \infty)$, for any x (marginal CDF of X);
- 2) $F_Y(y) = F_{XY}(\infty, y)$, for any y (marginal CDF of Y);
- 3) $F_{XY}(\infty, \infty) = 1$;

5.4 Joint Probability Density Function

Properties

4) $F_{XY}(-\infty, y) = F_{XY}(x, -\infty) = 0;$

5) $P(x_1 < X \leq x_2, y_1 < Y \leq y_2) =$
 $F_{XY}(x_2, y_2) - F_{XY}(x_1, y_2) - F_{XY}(x_2, y_1) + F_{XY}(x_1, y_1)$

5.4 Joint Probability Density Function

Example 5.4.1

Random variables X and Y have joint PDF

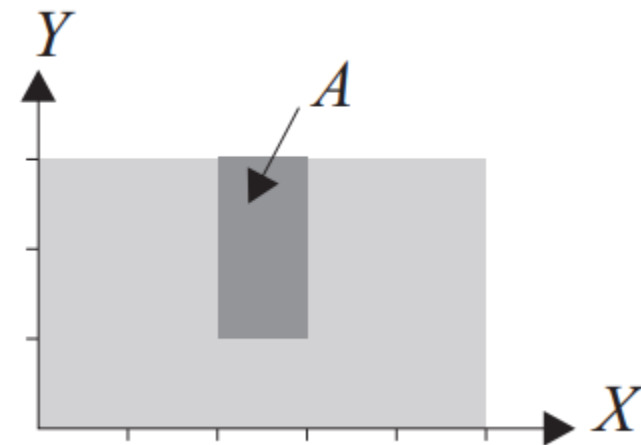
$$f_{X,Y}(x,y) = \begin{cases} c & 0 \leq x \leq 5, 0 \leq y \leq 3, \\ 0 & \text{otherwise.} \end{cases}$$

Find the constant c and $P[A] = P[2 \leq X < 3, 1 \leq Y < 3]$.

$$1 = \int_0^5 \int_0^3 c \, dy \, dx = 15c$$

$$c = 1/15$$

$$P[A] = \int_2^3 \int_1^3 \frac{1}{15} \, dv \, du = 2/15$$



5.4 Joint Probability Density Function

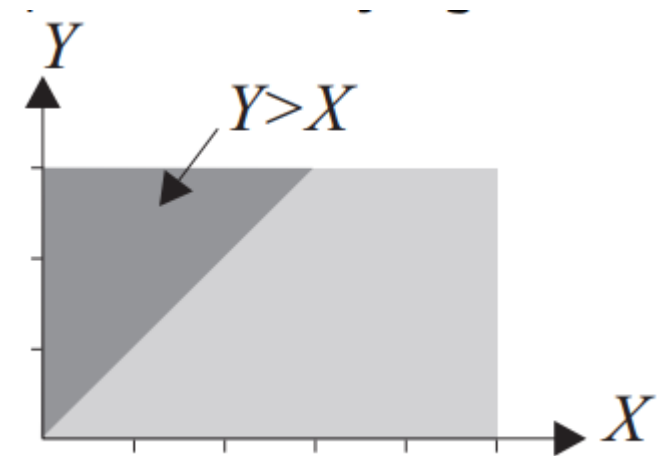
Example 5.4.2

random variables X and Y have joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 1/15 & 0 \leq x \leq 5, 0 \leq y \leq 3, \\ 0 & \text{otherwise.} \end{cases}$$

What is $P[A] = P[Y > X]$?

$$\begin{aligned} P[A] &= \int_0^3 \left(\int_x^3 \frac{1}{15} \right) dy dx \\ &= \int_0^3 \frac{3-x}{15} dx \\ &= -\frac{(3-x)^2}{30} \Big|_0^3 = \frac{3}{10} \end{aligned}$$



5.5 Marginal PDF

Definition

Marginal PDFs:

For **discrete** random variables:

$$P_X(x) = \sum_{y \in R_Y} P_{XY}(x, y).$$

For **continuous** random variables:

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y) dy, \quad \text{for all } x,$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x, y) dx, \quad \text{for all } y.$$

5.5 Marginal PDF

Example 5.5.1

The joint PDF of X and Y is

$$f_{X,Y}(x,y) = \begin{cases} 5y/4 & -1 \leq x \leq 1, x^2 \leq y \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

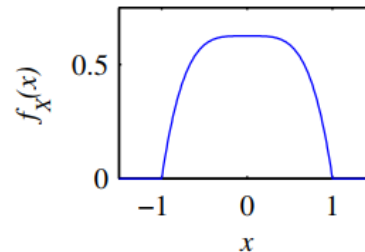
Find the marginal PDFs $f_X(x)$ and $f_Y(y)$.

When $x < -1$ or when $x > 1$, $f_{X,Y}(x,y) = 0$, and therefore $f_X(x) = 0$

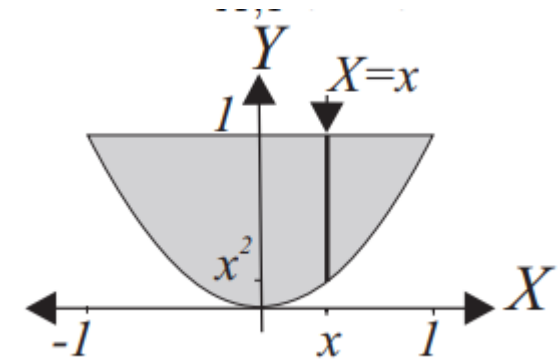
For $-1 \leq x \leq 1$

$$f_X(x) = \int_{x^2}^1 \frac{5y}{4} dy = \frac{5(1 - x^4)}{8}$$

The complete expression for the marginal PDF of X is



$$f_X(x) = \begin{cases} 5(1 - x^4)/8 & -1 \leq x \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$



5.5 Marginal PDF

Example 5.5.1

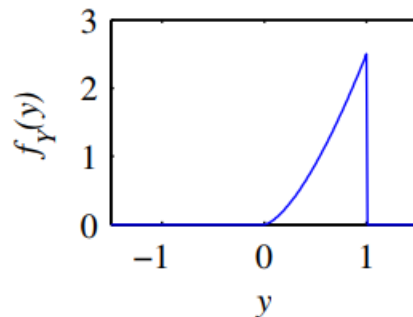
The joint PDF of X and Y is

$$f_{X,Y}(x,y) = \begin{cases} 5y/4 & -1 \leq x \leq 1, x^2 \leq y \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

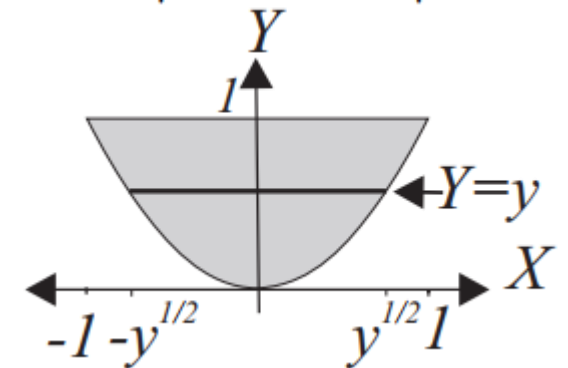
Find the marginal PDFs $f_X(x)$ and $f_Y(y)$.

$$y < 0 \text{ or } y > 1, f_Y(y) = 0 \quad f_Y(y) = \int_{-\sqrt{y}}^{\sqrt{y}} \frac{5y}{4} dx = \frac{5y}{4} x \Big|_{x=-\sqrt{y}}^{x=\sqrt{y}} = 5y^{3/2}/2$$

The complete marginal PDF of Y is



$$f_Y(y) = \begin{cases} (5/2)y^{3/2} & 0 \leq y \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$



5.5 Marginal PDF

Example 5.5.2

The joint probability density function of random variables X and Y is

$$f_{X,Y}(x,y) = \begin{cases} 6(x+y^2)/5 & 0 \leq x \leq 1, 0 \leq y \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Find $f_X(x)$ and $f_Y(y)$, the marginal PDFs of X and Y .

$$f_X(x) = 0 \text{ for } x < 0 \text{ or } x > 1$$

For $0 \leq x \leq 1$

$$f_X(x) = \frac{6}{5} \int_0^1 (x + y^2) dy = \frac{6}{5} (xy + y^3/3) \Big|_{y=0}^{y=1} = \frac{6x+2}{5}$$

$$f_X(x) = \begin{cases} (6x+2)/5 & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(y) = 0 \text{ for } y < 0 \text{ or } y > 1$$

For $0 \leq y \leq 1$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) dx = \frac{6}{5} \int_0^1 (x + y^2) dx = \frac{6}{5} \left(\frac{x^2}{2} + xy^2 \right) \Big|_{x=0}^{x=1} = \frac{6y^2+3}{5}.$$

$$f_Y(y) = \begin{cases} (3+6y^2)/5 & 0 \leq y \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$