

5.6 Independent Random Variables

Definition

Independent Random Variables:

Two random variables X and Y are independent if

$$P_{XY}(x_i, y_j) = P_X(x_i)P_Y(y_j), \quad \text{for all } x_i, y_j.$$

$$F_{XY}(x, y) = F_X(x)F_Y(y), \quad \text{for all } x, y.$$

5.6 Independent Random Variables

Example 5.6.1

$$f_{X,Y}(x,y) = \begin{cases} 4xy & 0 \leq x \leq 1, 0 \leq y \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Are X and Y independent?

The marginal PDFs of X and Y are

$$f_X(x) = \begin{cases} 2x & 0 \leq x \leq 1, \\ 0 & \text{otherwise,} \end{cases} \quad f_Y(y) = \begin{cases} 2y & 0 \leq y \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

$$f_{X,Y}(x,y) = f_X(x)f_Y(y)$$

X and Y are independent

5.7 Expected Value of a Function of Two Random Variables

Definition

Let X and Y are two random variables, and suppose that $Z = g(X, Y)$,
 $\Rightarrow Z$ is random variable.

$$E[g(X, Y)] = \sum_{(x_i, y_j) \in R_{XY}} g(x_i, y_j) P_{XY}(x_i, y_j)$$

$$E[g(X, Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f_{XY}(x, y) \cdot dx \cdot dy$$

$$E[a_1 g_1(X, Y) + \cdots + a_n g_n(X, Y)] = a_1 E[g_1(X, Y)] + \cdots + a_n E[g_n(X, Y)].$$

5.7 Expected Value of a Function of Two Random Variables

Theorms

$$E[a_1 g_1(X, Y) + \cdots + a_n g_n(X, Y)] = a_1 E[g_1(X, Y)] + \cdots + a_n E[g_n(X, Y)].$$

For any two random variables X and Y ,

$$E[X + Y] = E[X] + E[Y].$$

5.7 Expected Value of a Function of Two Random Variables

Theorms

The variance of the sum of two random variables is

$$\text{Var}[X + Y] = \text{Var}[X] + \text{Var}[Y] + 2 E[(X - \mu_X)(Y - \mu_Y)].$$

Since $E[X + Y] = \mu_X + \mu_Y$,

$$\begin{aligned}\text{Var}[X + Y] &= E[(X + Y - (\mu_X + \mu_Y))^2] \\ &= E[((X - \mu_X) + (Y - \mu_Y))^2] \\ &= E[(X - \mu_X)^2 + 2(X - \mu_X)(Y - \mu_Y) + (Y - \mu_Y)^2].\end{aligned}$$

We observe that each of the three terms in the preceding expected values is a function of X and Y . Therefore, Theorem 5.10 implies

$$\text{Var}[X + Y] = E[(X - \mu_X)^2] + 2 E[(X - \mu_X)(Y - \mu_Y)] + E[(Y - \mu_Y)^2].$$

The first and last terms are, respectively, $\text{Var}[X]$ and $\text{Var}[Y]$.

5.8 Covariance, Correlation and Independence

Theorms

The **covariance** between X and Y is defined as

$$\text{Cov}(X, Y) = E[(X - E[X])(Y - E[Y])] = E[XY] - (E[X])(E[Y]).$$

Proof:

$$\begin{aligned} E[(X - E[X])(Y - E[Y])] &= E[XY - X(E[Y]) - (E[X])Y + (E[X])(E[Y])] \\ &= E[XY] - (E[X])(E[Y]) - (E[X])(E[Y]) + (E[X])(E[Y]) \\ &= E[XY] - (E[X])(E[Y]). \end{aligned}$$

5.8 Covariance, Correlation and Independence

Properties

Lemma. The covariance has the following properties:

1) $\text{Cov}(X, X) = E[XX] - E[X]E[X] = E[X^2] - (E[X])^2 = \text{Var}(X).$

2) X & Y independent:

$$\text{Cov}(X, Y) = E[XY] - E[X]E[Y] = E[X]E[Y] - E[X]E[Y] = 0.$$

3) $\text{Cov}(X, Y) = \text{Cov}(Y, X)$

4) $\text{Cov}(aX, Y) = a\text{Cov}(X, Y) \quad a \in \mathbb{R}$

5.8 Covariance, Correlation and Independence

Properties

5) $\text{Cov}(X + c, Y) = \text{Cov}(X, Y)$

6) $\text{Cov}(X + Y, Z) = \text{Cov}(X, Z) + \text{Cov}(Y, Z)$

7) $\text{Cov}(X + Y, Z + W) = \text{Cov}(X, Z) + \text{Cov}(X, W) + \text{Cov}(Y, Z) + \text{Cov}(Y, W)$

$$\text{Cov}(2X + Y, 3Z + W) = 6\text{Cov}(X, Z) + 2\text{Cov}(X, W) + 3\text{Cov}(Y, Z) + \text{Cov}(Y, W)$$

5.8 Covariance, Correlation and Independence

Defention

Correlation Coefficient:

$$\rho_{XY} = \rho(X, Y) = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X) \text{Var}(Y)}} = \frac{\text{Cov}(X, Y)}{\sigma_X \sigma_Y}.$$

5.8 Covariance, Correlation and Independence

Properties

Properties of the correlation coefficient:

- 1) $-1 \leq \rho(X, Y) \leq 1$;
- 2) $\rho(aX + b, cY + d) = \rho(X, Y)$ for $a, c > 0$;
- 3) $\rho(X, Y) = 1$ if $Y = aX + b$ $a > 0$;
 $\rho(X, Y) = -1$ if $Y = aX + b$ $a < 0$.

5.8 Covariance, Correlation and Independence Properties

Consider two random variables X and Y :

- 1) If $\rho(X, Y) = 0$, we say that X and Y are **uncorrelated**.
- 2) If $\rho(X, Y) > 0$, we say that X and Y are **positively** correlated.
- 3) If $\rho(X, Y) < 0$, we say that X and Y are **negatively** correlated.

5.8 Covariance, Correlation and Independence

Defention

The correlation of X and Y is $r_{X,Y} = E[XY]$

Random variables X and Y are orthogonal if $r_{X,Y} = 0$.

Random variables X and Y are uncorrelated if $\text{Cov}[X, Y] = 0$.

5.8 Covariance, Correlation and Independence Properties

For independent random variables X and Y ,

$$(a) \ E[g(X)h(Y)] = E[g(X)] E[h(Y)],$$

$$(b) \ r_{X,Y} = E[XY] = E[X] E[Y],$$

$$(c) \ \text{Cov}[X, Y] = \rho_{X,Y} = 0,$$

$$(d) \ \text{Var}[X + Y] = \text{Var}[X] + \text{Var}[Y],$$

5.8 Covariance, Correlation and Independence

Example 5.8.1

Random variables L and T have joint PMF

$P_{L,T}(l, t)$	$t = 40 \text{ sec}$	$t = 60 \text{ sec}$
$l = 1 \text{ page}$	0.15	0.1
$l = 2 \text{ pages}$	0.30	0.2
$l = 3 \text{ pages}$	0.15	0.1.

Find the following quantities.

(a) $E[L]$ and $\text{Var}[L]$

$$E[L] = 1(0.25) + 2(0.5) + 3(0.25) = 2$$

$$E[L^2] = 1^2(0.25) + 2^2(0.5) + 3^2(0.25) = 4.5$$

$$\text{Var}[L] = E[L^2] - (E[L])^2 = 0.5$$

$P_{L,T}(l, t)$	$t = 40$	$t = 60$	$P_L(l)$
$l = 1$	0.15	0.1	0.25
$l = 2$	0.3	0.2	0.5
$l = 3$	0.15	0.1	0.25
$P_T(t)$	0.6	0.4	

5.8 Covariance, Correlation and Independence

Example 5.8.1

(b) $E[T]$ and $\text{Var}[T]$

$$E[T] = 40(0.6) + 60(0.4) = 48$$

$$E[T^2] = 40^2(0.6) + 60^2(0.4) = 2400$$

$$\text{Var}[T] = E[T^2] - (E[T])^2 = 96$$

$P_{L,T}(l, t)$	$t = 40$	$t = 60$	$P_L(l)$
$l = 1$	0.15	0.1	0.25
$l = 2$	0.3	0.2	0.5
$l = 3$	0.15	0.1	0.25
$P_T(t)$	0.6	0.4	

(c) The covariance $\text{Cov}[L, T]$

$$\text{Cov}[L, T] = E[LT] - E[L]E[T]$$

$$E[LT] = \sum_{t=40,60} \sum_{l=1}^3 ltP_{LT}(lt)$$

$$\begin{aligned} &= 1(40)(0.15) + 2(40)(0.3) + 3(40)(0.15) \\ &\quad + 1(60)(0.1) + 2(60)(0.2) + 3(60)(0.1) \\ &= 96. \end{aligned}$$

$$\text{Cov}[L, T] = E[LT] - E[L]E[T] = 96 - 2(48) = 0.$$

(d) The correlation coefficient $\rho_{L,T}$

Since $\text{Cov}[L, T] = 0$, the correlation coefficient is $\rho_{L,T} = 0$

5.8 Covariance, Correlation and Independence

Example 5.8.2

The joint probability density function of random variables X and Y is

$$f_{X,Y}(x,y) = \begin{cases} xy & 0 \leq x \leq 1, 0 \leq y \leq 2, \\ 0 & \text{otherwise.} \end{cases}$$

Find the following quantities.

(a) $E[X]$ and $\text{Var}[X]$

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) dy = \int_0^2 xy dy = \frac{1}{2}xy^2 \Big|_{y=0}^{y=2} = 2x.$$

$$f_X(x) = \begin{cases} 2x & 0 \leq x \leq 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) dx = \int_0^1 xy dx = \frac{1}{2}x^2y \Big|_{x=0}^{x=1} = \frac{y}{2}.$$

$$f_Y(y) = \begin{cases} y/2 & 0 \leq y \leq 2, \\ 0 & \text{otherwise.} \end{cases}$$

$$E[X] = \int_{-\infty}^{\infty} xf_X(x) dx = \int_0^1 2x^2 dx = \frac{2}{3} \quad E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x) dx = \int_0^1 2x^3 dx = \frac{1}{2}$$

$$\text{Var}[X] = E[X^2] - (E[X])^2 = \frac{1}{18}$$

5.8 Covariance, Correlation and Independence

Example 5.8.2

(b) $E[Y]$ and $\text{Var}[Y]$

$$E[Y] = \int_{-\infty}^{\infty} y f_Y(y) dy = \int_0^2 \frac{1}{2} y^2 dy = \frac{4}{3}$$

$$E[Y^2] = \int_{-\infty}^{\infty} y^2 f_Y(y) dy = \int_0^2 \frac{1}{2} y^3 dy = 2$$

$$\text{Var}[Y] = E[Y^2] - (E[Y])^2 = 2 - \frac{16}{9} = \frac{2}{9}$$

(c) The covariance $\text{Cov}[X, Y]$

$$\begin{aligned} E[XY] &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy f_{X,Y}(x, y) dx, dy \\ &= \int_0^1 \int_0^2 x^2 y^2 dx, dy = \left. \frac{x^3}{3} \right|_0^1 \left. \frac{y^3}{3} \right|_0^2 = \frac{8}{9} \end{aligned}$$

$$\text{Cov}[X, Y] = E[XY] - E[X]E[Y] = \frac{8}{9} - \frac{2}{3} \cdot \frac{4}{3} = 0$$

(d) The correlation coefficient $\rho_{X,Y}$

Since $\text{Cov}[X, Y] = 0$, the correlation coefficient is $\rho_{X,Y} = 0$

Selected Problems

All Questions of 5.1 – 5.8