

EELE 3340

Probability Theory



Chapter 7

Conditional Probability Models

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7.1 Conditioning a Random Variable by an Event

Definitions

Given the event B with $P[B] > 0$, the conditional cumulative distribution function of X is

$$F_{X|B}(x) = P[X \leq x|B].$$

Given the event B with $P[B] > 0$, the conditional probability mass function of X is

$$P_{X|B}(x) = P[X = x|B].$$

7.1 Conditioning a Random Variable by an Event

Definitions

For a random variable X and an event B with $P[B] > 0$, the conditional PDF of X given B is

$$f_{X|B}(x) = \frac{dF_{X|B}(x)}{dx}.$$

7.1 Conditioning a Random Variable by an Event

Theorem

For a random variable X and an event $B \subset S_X$ with $P[B] > 0$, the conditional PDF of X given B is

$$\text{Discrete: } P_{X|B}(x) = \begin{cases} \frac{P_X(x)}{P[B]} & x \in B, \\ 0 & \text{otherwise} \end{cases}$$

$$\text{Continuous: } f_{X|B}(x) = \begin{cases} \frac{f_X(x)}{P[B]} & x \in B, \\ 0 & \text{otherwise.} \end{cases}$$

7.1 Conditioning a Random Variable by an Event

Example 7.1.1

A website distributes instructional videos on bicycle repair. The length of a video in minutes X has PMF

$$P_X(x) = \begin{cases} 0.15 & x = 1, 2, 3, 4, \\ 0.1 & x = 5, 6, 7, 8, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose the website has two servers, one for videos shorter than five minutes and the other for videos of five or more minutes. What is the PMF of video length in the second server?

$$P_{X|L}(x) = \begin{cases} \frac{P_X(x)}{P[L]} & x = 5, 6, 7, 8, \\ 0 & \text{otherwise.} \end{cases} \quad P[L] = \sum_{x=5}^8 P_X(x) = 0.4$$

$$P_{X|L}(x) = \begin{cases} 0.1/0.4 = 0.25 & x = 5, 6, 7, 8, \\ 0 & \text{otherwise.} \end{cases}$$

7.1 Conditioning a Random Variable by an Event

Example 7.1.2

Suppose X , the time in integer minutes you wait for a bus, has the discrete uniform PMF

$$P_X(x) = \begin{cases} 1/20 & x = 1, 2, \dots, 20, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose the bus has not arrived by the eighth minute; what is the conditional PMF of your waiting time X ?

Let A denote the event $X > 8$

$$P[A] = \underline{12/20}$$

$$P_{X|X>8}(x) = \begin{cases} \frac{1/20}{12/20} = \frac{1}{12} & x = 9, 10, \dots, 20, \\ 0 & \text{otherwise.} \end{cases}$$

7.1 Conditioning a Random Variable by an Event

Theorem

For random variable X resulting from an experiment with partition

$$B_1, \dots, B_m,$$

Discrete: $P_X(x) = \sum_{i=1}^m \underline{P_{X|B_i}(x) P[B_i]}$

Continuous: $f_X(x) = \sum_{i=1}^m f_{X|B_i}(x) P[B_i]$

7.1 Conditioning a Random Variable by an Event

Example 7.1.3

Random variable X is a voltage at the receiver of a modem. When symbol "0" is transmitted (event B_0), X is the Gaussian $(-5, 2)$ random variable. When symbol "1" is transmitted (event B_1), X is the Gaussian $(5, 2)$ random variable. Given that symbols "0" and "1" are equally likely to be sent, what is the PDF of X ?

$$P[B_0] = P[B_1] = 1/2$$

$$f_{X|B_0}(x) = \frac{1}{2\sqrt{2\pi}} e^{-(x+5)^2/8}$$

$$f_{X|B_1}(x) = \frac{1}{2\sqrt{2\pi}} e^{-(x-5)^2/8}$$

$$\underline{f_X(x) = f_{X|B_0}(x) P[B_0] + f_{X|B_1}(x) P[B_1]} = \frac{1}{4\sqrt{2\pi}} \left(e^{-(x+5)^2/8} + e^{-(x-5)^2/8} \right)$$

7.1 Conditioning a Random Variable by an Event

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$$P[B_0] = P[B_1] = 1/2$$

$$f_{X|B_0}(x) = \frac{1}{2\sqrt{2\pi}} e^{-(x+5)^2/8}$$

$$f_{X|B_1}(x) = \frac{1}{2\sqrt{2\pi}} e^{-(x-5)^2/8}$$

$$f_X(x) = f_{X|B_0}(x) P[B_0] + f_{X|B_1}(x) P[B_1] = \frac{1}{4\sqrt{2\pi}} \left(e^{-(x+5)^2/8} + e^{-(x-5)^2/8} \right)$$

7.2 Conditional Expected Value Given an Event

Theorem

Discrete X :

- (a) For any $x \in B$, $P_{X|B}(x) \geq 0$.
- (b) $\sum_{x \in B} P_{X|B}(x) = 1$.
- (c) The conditional probability that X is in the set C is

$$P[C|B] = \sum_{x \in C} P_{X|B}(x).$$

Continuous X :

- (a) For any $x \in B$, $f_{X|B}(x) \geq 0$.
- (b) $\int_B f_{X|B}(x) dx = 1$.
- (c) The conditional probability that X is in the set C is

$$P[C|B] = \int_C f_{X|B}(x) dx.$$

7.2 Conditional Expected Value Given an Event

Definition

The conditional expected value of random variable X given condition B is

$$\text{Discrete: } E[X|B] = \sum_{x \in B} x P_{X|B}(x)$$

$$\text{Continuous: } E[X|B] = \int_{-\infty}^{\infty} x f_{X|B}(x) dx.$$

7.2 Conditional Expected Value Given an Event

Theorems

For a random variable X resulting from an experiment with partition $B_1, \dots, B_m$,

$$E[X] = \sum_{i=1}^m E[X|B_i] P[B_i].$$

The conditional expected value of $Y = g(X)$ given condition B is

Discrete: $E[Y|B] = E[g(X)|B] = \sum_{\underline{x \in B}} g(x) P_{X|B}(x)$

Continuous: $E[Y|B] = E[g(X)|B] = \underline{\int_{-\infty}^{\infty} g(x) f_{X|B}(x) dx.}$

7.2 Conditional Expected Value Given an Event

Definition

The conditional variance of X given event B is

$$\text{Var}[X|B] = \mathbb{E} \left[(X - \mu_{X|B})^2 | B \right] = \mathbb{E} [X^2 | B] - \mu_{X|B}^2.$$

The conditional standard deviation is $\sigma_{X|B} = \sqrt{\text{Var}[X|B]}$.

7.2 Conditional Expected Value Given an Event

Example 7.2.1

On the Internet, data is transmitted in packets. In a simple model for World Wide Web traffic, the number of packets N needed to transmit a Web page depends on whether the page has graphic images. If the page has images (event I), then N is uniformly distributed between 1 and 50 packets. If the page is just text (event T), then N is uniform between 1 and 5 packets. Assuming a page has images with probability $1/4$, find the

(a) conditional PMF $P_{N|I}(n)$

$$P_{N|I}(n) = \begin{cases} 0.02 & n = 1, 2, \dots, 50, \\ 0 & \text{otherwise.} \end{cases}$$

(b) conditional PMF $P_{N|T}(n)$

$$P_{N|T}(n) = \begin{cases} 0.2 & n = 1, \dots, 5, \\ 0 & \text{otherwise.} \end{cases}$$

(c) PMF $P_N(n)$

$$P[T] = 1 - P[I] = \underline{3/4}$$

$$P_N(n) = \underline{P_{N|T}(n)} \underline{P[T]} + \underline{P_{N|I}(n)} \underline{P[I]} = \begin{cases} 0.155 & n = 1, \dots, 5, \\ 0.005 & n = 6, \dots, 50, \\ 0 & \text{otherwise.} \end{cases}$$

(d) conditional PMF $P_{N|N \leq 10}(n)$

$$P[N \leq 10] = \sum_{n=1}^{10} P_N(n) = \underline{(0.155)(5)} + \underline{(0.005)(5)} = 0.80$$

$$P_{N|N \leq 10}(n) = \begin{cases} \frac{P_N(n)}{P[N \leq 10]} & n \leq 10, \\ 0 & \text{otherwise,} \end{cases} = \begin{cases} \frac{0.155}{0.8} = 0.19375 & n = 1, \dots, 5, \\ \frac{0.005}{0.8} = 0.00625 & n = 6, \dots, 10, \\ 0 & \text{otherwise.} \end{cases}$$

7.2 Conditional Expected Value Given an Event

Example 7.2.1

On the Internet, data is transmitted in packets. In a simple model for World Wide Web traffic, the number of packets N needed to transmit a Web page depends on whether the page has graphic images. If the page has images (event I), then N is uniformly distributed between 1 and 50 packets. If the page is just text (event T), then N is uniform between 1 and 5 packets. Assuming a page has images with probability $1/4$, find the

$$E[N|N \leq 10]$$

$$\begin{aligned} E[N|N \leq 10] &= \sum_n n P_{N|N \leq 10}(n) \\ &= \sum_{n=1}^5 0.19375n + \sum_{n=6}^{10} 0.00625n = 3.15625 \end{aligned}$$

$$\text{Var}[N|N \leq 10]$$

$$\begin{aligned} E[N^2|N \leq 10] &= \sum_n \underline{n^2 P_{N|N \leq 10}(n)} = \sum_{n=1}^5 0.19375n^2 + \sum_{n=6}^{10} 0.00625n^2 \\ &= 0.19375(55) + 0.00625(330) = 12.719 \\ \text{Var}[N|N \leq 10] &= E[N^2|N \leq 10] - (E[N|N \leq 10])^2 \\ &= 12.719 - (3.156)^2 = 2.757 \end{aligned}$$

7.2 Conditional Expected Value Given an Event

Example 7.2.2

Y is a continuous uniform (0, 10) random variable. Find the following:

(a) $P[Y \leq 6]$

$$f_Y(y) = \begin{cases} 1/10 & 0 < y < 10, \\ 0 & \text{otherwise.} \end{cases}$$

$$P[Y \leq 6] = \int_{-\infty}^6 f_Y(y) dy = \int_0^6 (1/10) dy = 0.6$$

(b) the conditional PDF $f_{Y|Y \leq 6}(y)$

$$f_{Y|Y \leq 6}(y) = \begin{cases} \frac{f_Y(y)}{P[Y \leq 6]} & y \leq 6, \\ 0 & \text{otherwise,} \end{cases} = \begin{cases} 1/6 & 0 \leq y \leq 6, \\ 0 & \text{otherwise.} \end{cases}$$

7.2 Conditional Expected Value Given an Event

Example 7.2.2

Y is a continuous uniform $(0, 10)$ random variable. Find the following:

(c) $P[Y > 8]$

$$P[Y > 8] = \int_8^{10} \frac{1}{10} dy = \underline{0.2}$$

(d) the conditional PDF $f_{Y|Y>8}(y)$

$$f_{Y|Y>8}(y) = \begin{cases} \frac{f_Y(y)}{P[Y>8]} & y > 8, \\ 0 & \text{otherwise,} \end{cases} = \begin{cases} \frac{1}{2} & 8 < y \leq 10, \\ 0 & \text{otherwise.} \end{cases}$$

7.2 Conditional Expected Value Given an Event

Example 7.2.2

Y is a continuous uniform $(0, 10)$ random variable. Find the following:

$$E[Y|Y \leq 6]$$

$$E[Y|Y \leq 6] = \int_{-\infty}^{\infty} y f_{Y|Y \leq 6}(y) dy = \int_0^6 \frac{y}{6} dy = 3$$

$$\text{Var}[Y|Y \leq 6]$$

Selected Problems

Solve all questions of 7.1 and 7.2