

EELE 3340

Probability Theory



Chapter 13

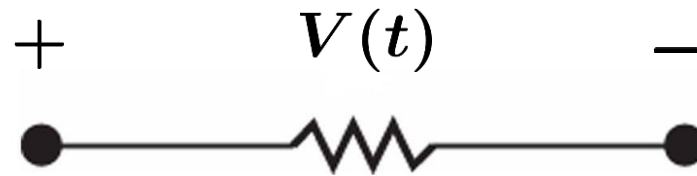
Stochastic (Random) Processes

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Random process

Definitions

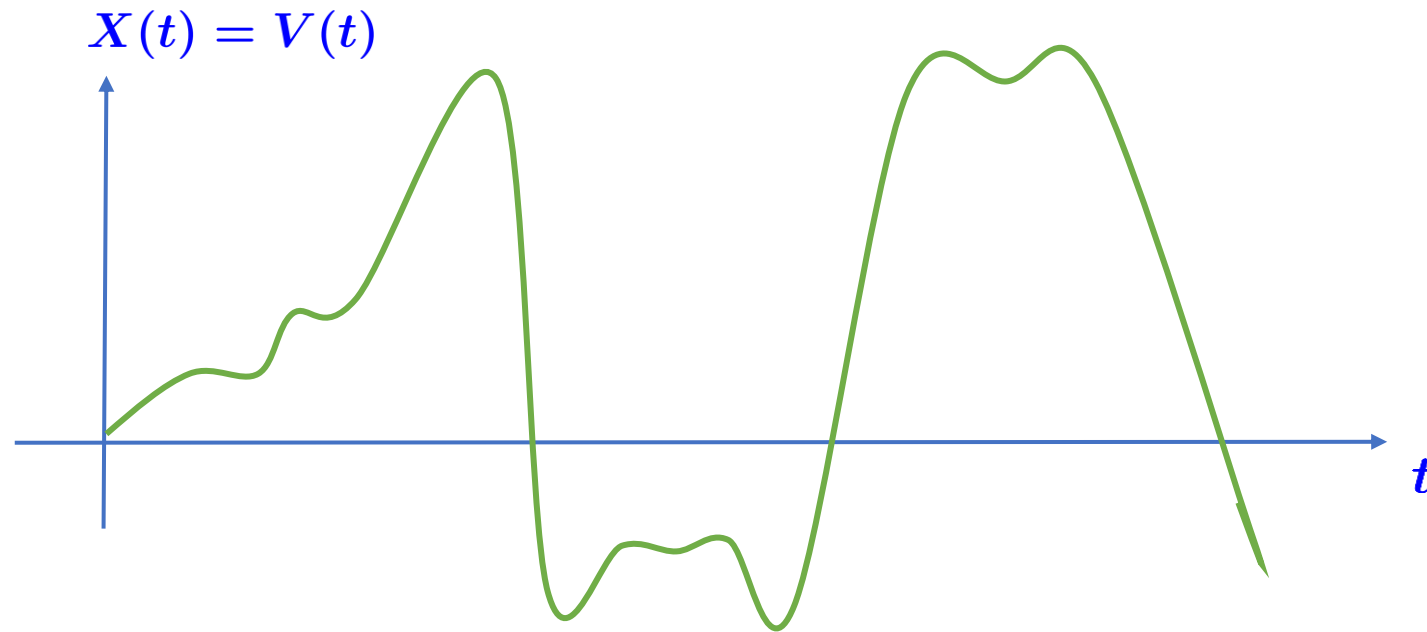
- Random process (stochastic process)
A family of random variable (usually infinite)
- Example. Many random phenomena that occur in nature are function of time.



Random process

Definitions

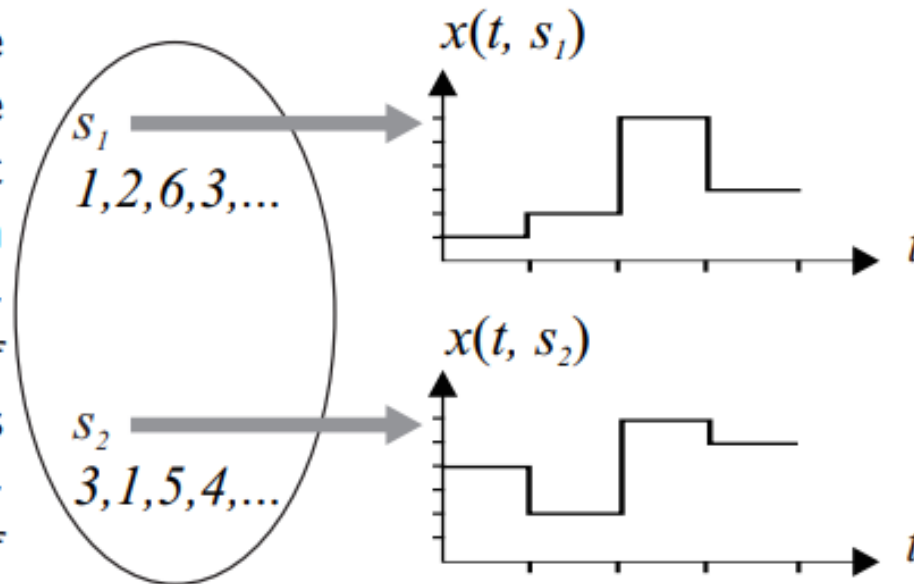
The thermal noise voltage generated in a resistor



Random process

Definitions

Suppose that at time instants $T = 0, 1, 2, \dots$, we roll a die and record the outcome N_T where $1 \leq N_T \leq 6$. We then define the random process $X(t)$ such that for $T \leq t < T + 1$, $X(t) = N_T$. In this case, the experiment consists of an infinite sequence of rolls and a sample function is just the waveform corresponding to the particular sequence of rolls. This mapping is depicted on the right.

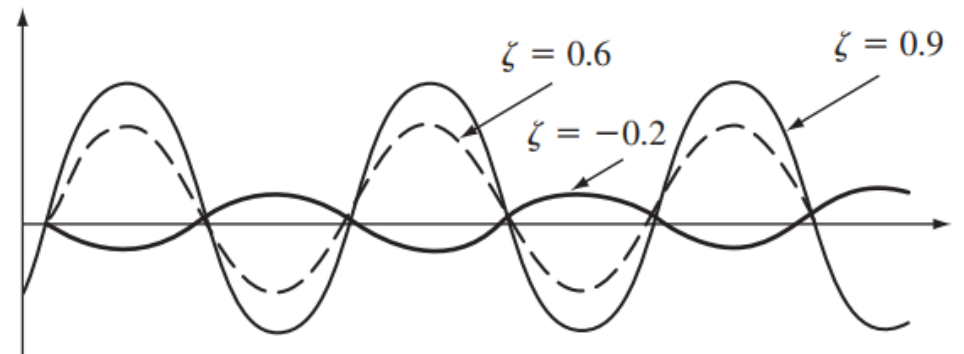


Random process

Definitions

Let ζ be selected at random from the interval $[-1, 1]$. Define the continuous-time random process $X(t, \zeta)$ by

$$X(t, \zeta) = \zeta \cos(2\pi t) \quad -\infty < t < \infty.$$



Random process

Definitions

Discrete-Value and Continuous-Value Processes

$X(t)$ is a discrete-value process if the set of all possible values of $X(t)$ at all times t is a countable set S_X ; otherwise $X(t)$ is a continuous-value process.

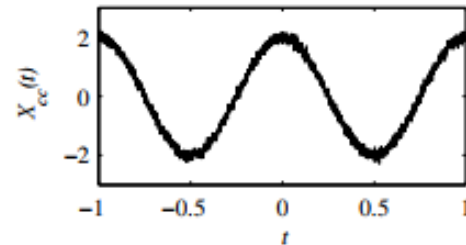
Discrete-Time and Continuous-Time Processes

The stochastic process $X(t)$ is a discrete-time process if $X(t)$ is defined only for a set of time instants, $t_n = nT$, where T is a constant and n is an integer; otherwise $X(t)$ is a continuous-time process.

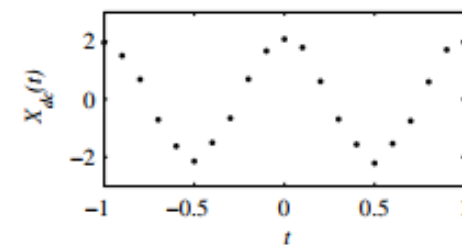
Random process

Definitions

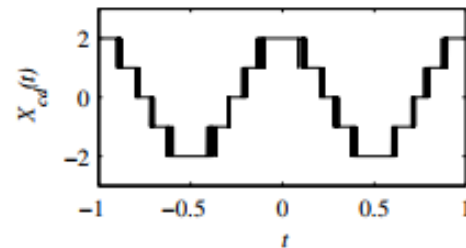
Continuous-Time, Continuous-Value



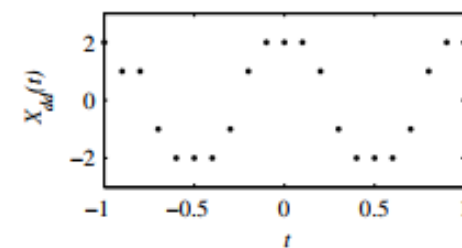
Discrete-Time, Continuous-Value



Continuous-Time, Discrete-Value



Discrete-Time, Discrete-Value



Sample functions of four kinds of stochastic processes. $X_{cc}(t)$ is a continuous-time, continuous-value process. $X_{dc}(t)$ is discrete-time, continuous-value process obtained by sampling $X_{cc}(t)$ every 0.1 seconds. Rounding $X_{cc}(t)$ to the nearest integer yields $X_{cd}(t)$, a continuous-time, discrete-value process. Lastly, $X_{dd}(t)$, a discrete-time, discrete-value process, can be obtained either by sampling $X_{cd}(t)$ or by rounding $X_{dc}(t)$.

Mean and Correlation Functions

For a random process $\{X(t), t \in J\}$ the **mean function** $\mu_X(t) : J \rightarrow \mathbb{R}$, is defined as

$$\mu_X(t) = E[X(t)].$$

For a random process $\{X(t), t \in J\}$, the **autocorrelation function** or, simply, the **correlation function**, $R_X(t_1, t_2)$, is defined by

$$R_X(t_1, t_2) = E[X(t_1)X(t_2)], \quad \text{for } t_1, t_2 \in J.$$

Mean and Correlation Functions

For a random process $\{X(t), t \in J\}$, the **autocovariance function** or, simply, the **covariance function**, $C_X(t_1, t_2)$, is defined by

$$\begin{aligned} C_X(t_1, t_2) &= \text{Cov}(X(t_1), X(t_2)) \\ &= R_X(t_1, t_2) - \mu_X(t_1)\mu_X(t_2), \quad \text{for } t_1, t_2 \in J. \end{aligned}$$

Mean and Correlation Functions

Example 9.1

Let $X(t) = A \cos 2\pi t$, where A is some random variable. The mean of $X(t)$ is

$$m_X(t) = E[A \cos 2\pi t] = E[A] \cos 2\pi t.$$

The autocorrelation is

$$\begin{aligned} R_X(t_1, t_2) &= E[A \cos 2\pi t_1 A \cos 2\pi t_2] \\ &= E[A^2] \cos 2\pi t_1 \cos 2\pi t_2, \end{aligned}$$

and the autocovariance is then

$$\begin{aligned} C_X(t_1, t_2) &= R_X(t_1, t_2) - m_X(t_1)m_X(t_2) \\ &= \{E[A^2] - E[A]^2\} \cos 2\pi t_1 \cos 2\pi t_2 \\ &= \text{VAR}[A] \cos 2\pi t_1 \cos 2\pi t_2. \end{aligned}$$

Mean and Correlation Functions

Example 9.2

Let $X(t) = \cos(\omega t + \Theta)$, where Θ is uniformly distributed in the interval $(-\pi, \pi)$

The mean of $X(t)$ is :

$$m_X(t) = E[\cos(\omega t + \Theta)] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(\omega t + \theta) d\theta = 0.$$

The autocorrelation and autocovariance are then

$$\begin{aligned} C_X(t_1, t_2) &= R_X(t_1, t_2) = E[\cos(\omega t_1 + \Theta) \cos(\omega t_2 + \Theta)] \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{2} \{ \cos(\omega(t_1 - t_2)) + \cos(\omega(t_1 + t_2) + 2\theta) \} d\theta \\ &= \frac{1}{2} \cos(\omega(t_1 - t_2)), \end{aligned}$$

where we used the identity $\cos(a) \cos(b) = 1/2 \cos(a + b) + 1/2 \cos(a - b)$. Note that $m_X(t)$ is a constant and that $C_X(t_1, t_2)$ depends only on $|t_1 - t_2|$. Note as well that the samples at time t_1 and t_2 are uncorrelated if $\omega(t_1 - t_2) = k\pi$ where k is any integer.

Mean and Correlation Functions

Example 9.3

Let $X(t) = \cos(\omega t + \Theta)$ and $Y(t) = \sin(\omega t + \Theta)$, where Θ is a random variable uniformly distributed in $[-\pi, \pi]$. Find the cross-covariance of $X(t)$ and $Y(t)$.

$X(t)$ and $Y(t)$ are zero mean.

$$\begin{aligned} C_{X,Y}(t_1, t_2) &= R_{X,Y}(t_1, t_2) = E[\cos(\omega t_1 + \Theta) \sin(\omega t_2 + \Theta)] \\ &= E\left[-\frac{1}{2}\sin(\omega(t_1 - t_2)) + \frac{1}{2}\sin(\omega(t_1 + t_2) + 2\Theta)\right] \\ &= -\frac{1}{2}\sin(\omega(t_1 - t_2)) \end{aligned}$$

Stationary Processes

A continuous-time random process $\{X(t), t \in \mathbb{R}\}$ is **strict-sense stationary** or simply **stationary** if, for all $t_1, t_2, \dots, t_r \in \mathbb{R}$ and all $\Delta \in \mathbb{R}$, the joint CDF of

$$X(t_1), X(t_2), \dots, X(t_r)$$

is the same as the joint CDF of

$$X(t_1 + \Delta), X(t_2 + \Delta), \dots, X(t_r + \Delta).$$

The mean and variance of the first-order stationary random process are independent of time.

Stationary Processes

A discrete-time random process $\{X(n), n \in \mathbb{Z}\}$ is **strict-sense stationary** or simply **stationary**, if for all $n_1, n_2, \dots, n_r \in \mathbb{Z}$ and all $D \in \mathbb{Z}$, the joint CDF of

$$X(n_1), X(n_2), \dots, X(n_r)$$

is the same as the joint CDF of

$$X(n_1 + D), X(n_2 + D), \dots, X(n_r + D).$$