



## Final Exam

| QUESTIONS    | MARKS |
|--------------|-------|
| Ques. 1 (10) |       |
| Ques. 2 (8)  |       |
| Ques. 3 (6)  |       |
| Ques. 4 (6)  |       |
| Ques. 5 (10) |       |
| Ques. 6 (10) |       |
| Total (50)   |       |

Time Allowed: 120 minutes

Student Name:.....

Student ID:.....

### Question (1): Answer the following questions.

- a) What is the probability of 10 people in a room that no two of them have a birthday in the same month?

*Solved in Mid.*

- b) In a die game, each person rolls a fair die which has the numbers 1 to 6 on it. The Player wins if the die shows up as a 'doubles' (such as 3-3 or 6-6) or if the sum of the die is 7. What is the probability that the player wins?

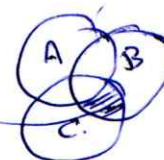
*Solved in lecture*

- c) Suppose now that events A, B and C are mutually independent with  $P(A) = 0.3$ ,  $P(B) = 0.4$ ,  $P(C) = 0.5$ . Compute  $P(A^c \cap B \cap C)$ .

$$P(A \cap B \cap C) = 0.06$$

$$P(A^c \cap B \cap C) = P(B \cap C) - P(A \cap B \cap C)$$

$$= 0.4 \times 0.5 - 0.06 = 0.14$$

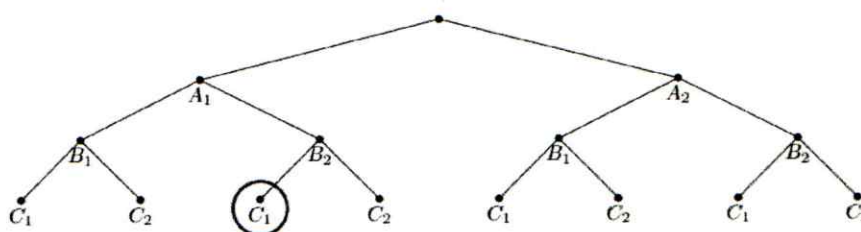


- d) In a binary communication channel, it is equally likely to send either a 1 or a 0. The probability of error given that a 1 was sent is  $1/9$ , while the probability of an error given that a 0 was sent is  $2/9$ . What is the probability that a 1 is received?

*Solved in med.*

- e) For the following tree, which does represent the probability of the circled node?

A.  $C_1$    B.  $B_2 \cap C_1$    C.  $A_1 \cap B_2 \cap C_1$    D.  $C_1 | B_2 \cap A_1$ .



## QUESTION (2):

- A) The number of hits at a website per second is a Poisson random variable with mean 10 hits/second. Using the central limit theorem, estimate the probability that more than 650 hits occurred in one minute. (Hint: For a Poisson random variable  $X$ ,  $E[X] = \text{Var}(X) = \alpha$  (Poisson Parameter))

Solved In lecture.

- B) Let  $\{X_n, n \in \mathbb{Z}\}$  be a discrete-time random process, defined as

$$X_n = 2 \cos\left(\frac{\pi n}{8} + \Phi\right),$$

where  $\Phi \sim \text{Uniform}(0, 2\pi)$

- 1) Find the mean function  $\mu_X(n)$ .
- 2) Find the correlation function  $R_X(m, n)$ .

$$f_\Phi = \begin{cases} \frac{1}{2\pi} & 0 \leq \Phi \leq 2\pi \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} \textcircled{1} \quad \mu_X &= 2 \int_{-\infty}^{\infty} \cos\left(\frac{\pi n}{8} + \phi\right) \cdot \frac{1}{2\pi} d\phi \\ &= \frac{1}{\pi} \int_0^{2\pi} \cos\left(\frac{\pi n}{8} + \phi\right) d\phi = 0. \end{aligned}$$

$$\begin{aligned} \textcircled{2} \quad R_X(m, n) &= E\left[2 \cos\left(\frac{m\pi}{8} + \phi\right) \cdot 2 \cos\left(\frac{n\pi}{8} + \phi\right)\right] \\ &= 4 E\left[\cos\left(\frac{\pi(m+n)}{8} + 2\phi\right) - \cos\left(\frac{(m-n)\pi}{8}\right)\right] \\ &= 4 \cos\left[\frac{(m-n)\pi}{8}\right] \end{aligned}$$

### QUESTION (3):

A) Any student needs 0.55 to pass a test graded on the scale 0 to 1. Student scores are modeled by the following probability density function:

$$f(x) = \begin{cases} Ax & 0 \leq x \leq 0.5 \\ 4 - 4x & 0.5 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Where  $A$  is a constant. Find:

- 1)  $A$ .
- 2) The probability that a random student passes the exam.

$$\textcircled{1} \int_{-\infty}^{\infty} f(x) dx = 1 \Rightarrow \int_0^{0.5} Ax dx + \int_{0.5}^1 (4 - 4x) dx = 1$$
$$\Rightarrow \boxed{A = 4}$$

$$\textcircled{2} P(X > 0.55) = \int_{0.55}^1 (4 - 4x) dx = \underline{\underline{0.405}}$$

B) Toss a fair coin 3 times. Let  $X$  = the number of heads on the first toss,  $Y$  = the total number of heads on the last two tosses.

- 1) Give the joint probability mass function table for  $X$  and  $Y$ .
- 2) Compute  $\text{Cov}(X, Y)$ . Are  $X$  and  $Y$  independent?

$$\textcircled{1} P_X = \begin{cases} \frac{1}{2} & X = 0 \\ \frac{1}{2} & X = 1 \end{cases} \quad P_Y = \begin{cases} \frac{1}{4} & Y = 0 \\ \frac{1}{2} & Y = 1 \\ \frac{1}{4} & Y = 2 \end{cases}$$

| $X \backslash Y$ | 0             | 1             | 2             |               |
|------------------|---------------|---------------|---------------|---------------|
| 0                | $\frac{1}{8}$ | $\frac{1}{4}$ | $\frac{1}{8}$ | $\frac{1}{2}$ |
| 1                | $\frac{1}{8}$ | $\frac{2}{8}$ | $\frac{1}{8}$ | $\frac{1}{2}$ |
|                  | $\frac{1}{4}$ | $\frac{1}{2}$ | $\frac{1}{4}$ |               |

$$\textcircled{2} E_X = \frac{1}{2} \quad E_Y = 1$$

#### QUESTION (4):

Consider two random variables  $X$  and  $Y$  with joint PMF given in the following table.

|         | $Y = 1$        | $Y = 2$        |
|---------|----------------|----------------|
| $X = 1$ | $\frac{1}{3}$  | $\frac{1}{12}$ |
| $X = 2$ | $\frac{1}{6}$  | 0              |
| $X = 4$ | $\frac{1}{12}$ | $\frac{1}{3}$  |

[A] Find  $P(X \leq 2, Y > 1)$ .

[B] Find the marginal PMFs of  $X$  and  $Y$ .

[C] Find  $P(Y=2|X=1)$ .

[D] Find  $E[X]$ .

$$\textcircled{A} \quad p(2,2) + p(1,2) = 0 + \frac{1}{12} = \boxed{\frac{1}{12}}.$$

$$\textcircled{B} \quad P_X = \begin{cases} \frac{5}{12} & x=1 \\ \frac{2}{12} & x=2 \\ \frac{5}{12} & x=3 \end{cases} \quad \bigg| \quad P_Y = \begin{cases} \frac{7}{12} & y=1 \\ \frac{5}{12} & y=2 \end{cases}$$

$$\textcircled{C} \quad P(Y=2|X=1) = \frac{p(2,1)}{p(X=1)} = \frac{\frac{1}{12}}{\frac{5}{12}} = \boxed{\frac{1}{5}}.$$

$$\textcircled{D} \quad E_X = \frac{5}{12} + \frac{8}{12} + \frac{20}{12} = \boxed{\frac{29}{12}}$$

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**QUESTION (5):**

Let  $X$  and  $Y$  be two jointly continuous random variables with joint PDF

$$f_{XY}(x, y) = \begin{cases} \frac{1}{4}x^2 + \frac{1}{6}y & -1 \leq x \leq 1, 0 \leq y \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the marginal PDFs,  $f_X(x)$  and  $f_Y(y)$ .

(b) Find  $P(X > 0, Y < 1)$ .

(c) Find  $P(X > 0 \text{ or } Y < 1)$ .

(d) Find  $P(X > 0 | Y < 1)$ .

$$\text{a)} \quad f_x = \int_0^2 \left( \frac{1}{4}x^2 + \frac{1}{6}y \right) dy = \frac{1}{2}x^2 + \frac{1}{3}$$

$$f_y = \int_{-1}^1 \left( \frac{1}{4}x^2 + \frac{1}{6}y \right) dx = \frac{1}{6} + \frac{1}{3}y$$

$$\text{b)} \quad P(X > 0, Y < 1) = \int_0^1 \int_0^1 \left( \frac{1}{4}x^2 + \frac{1}{6}y \right) dx dy = \left[ \frac{1}{6} \right]$$

$$\begin{aligned} \text{c)} \quad P(X > 0 \text{ or } Y < 1) &= 1 - P(X < 0 \text{ \& } Y > 1) \\ &= 1 - \int_{-1}^0 \int_1^2 \left( \frac{1}{4}x^2 + \frac{1}{6}y \right) dy dx \\ &= \left[ \frac{2}{3} \right] \end{aligned}$$

$$\begin{aligned} \text{d)} \quad P(X > 0 | Y < 1) &= \frac{P(X > 0, Y < 1)}{P(Y < 1)} \\ &= \frac{\frac{1}{6}}{\int_0^1 \left( \frac{1}{6} + \frac{1}{3}y \right) dy} = \left[ \frac{1}{2} \right] \end{aligned}$$

### QUESTION (6):

The average life of an electronic device follows a Gaussian distribution with average of 10 years with a standard deviation of 3 years. If the device fails while under guarantee, the manufacturer replaces it free.

- What fraction of the devices will be functional for more than 10 years?
- What fraction of the devices will be functional for less than 3 year?
- If the manufacturer plans to replace only 3% of the devices that could fail, how long the manufacturer should offer the guarantee to be?

$$\mu = 10 \quad \text{et} \quad \sigma = 3.$$

$$\begin{aligned} \text{[a]} \quad P(X > 10) &= P\left(\frac{X - \mu}{\sigma} > \frac{10 - \mu}{\sigma}\right) \\ &= P(Z > 0) = 0.5. \end{aligned}$$

$$\begin{aligned} \text{[b]} \quad P(X < 3) &= P(Z < -2.33) \\ &= 1 - \Phi(2.33) \\ &= \underline{\underline{0.001}}. \end{aligned}$$

$$\text{[c]} \quad 1 - \Phi(A) = 0.03.$$

$$\Phi(A) = 0.97.$$

$$A = -1.88 = \frac{X - 10}{3}.$$

$$\Rightarrow X = 4.3 \text{ years}$$