

Probability - Final Exam

Question (1):

$$[a] \quad P(C) = 0.25, \quad P(D) = 0.45, \quad P(C \cap D) = 0.1$$

$$\begin{aligned} \Rightarrow P(C \cup D) &= P(C) + P(D) - P(C \cap D) \\ &= 0.25 + 0.45 - 0.1 \\ &= 0.6 \end{aligned}$$

غير متوافق
في الكل

$$\Rightarrow P(C^c \cap D) = P(D) - P(C \cap D) = 0.45 - 0.10 = 0.35$$

$$\Rightarrow \text{So, } P(C^c \cap D) = \boxed{0.35}$$

$$[b] \quad \begin{aligned} (\text{Doubles}) &= \{(1,1), (2,2), (3,3), (4,4), (5,5), (6,6)\} \\ (\text{Sum of 7}) &= \{(1,6), (6,1), (2,5), (5,2), (3,4), (4,3)\} \end{aligned}$$

12
elements

$$\Rightarrow \text{Probability of wins} = \frac{12}{36} = \frac{1}{3} = \boxed{0.33}$$

$$[c] \quad P(A \cup B) = 1 - P(A \cup B)^c = 1 - P(A^c \cap B^c) = 1 - \frac{2}{3} = \boxed{\frac{1}{3}}$$

$$[d] \quad \begin{aligned} P(A^c \cap B \cap C) &= P(A^c) P(B) P(C) = (1 - P(A)) P(B) P(C) = (1 - 0.3)(0.4)(0.5) \\ P(A^c \cap B \cap C) &= \boxed{0.14} \end{aligned}$$

$$[e] \quad \boxed{P(B_2 | A_1)} \Rightarrow \text{Choice } \boxed{C}$$

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Question (2):

$$f_X(x) = \begin{cases} 2x, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases} \quad f_Y(y) = \begin{cases} 3y^2, & 0 \leq y \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

a) $E[XY] = E[X] E[Y]$

$$\Rightarrow E[X] = \int_0^1 2x^2 dx \Rightarrow E[X] = \frac{2}{3} = 0.67$$

$$\Rightarrow E[Y] = \int_0^1 3y^3 dy \Rightarrow E[Y] = \frac{3}{4} = 0.75$$

$$\Rightarrow E[XY] = E[X] E[Y] = \frac{2}{3} \times \frac{3}{4} = \frac{1}{2} \Rightarrow E[XY] = \boxed{\frac{1}{2}}$$

b) $\text{Cov}(X, Y) = E[XY] - (E[X])(E[Y]) = \frac{1}{2} - \left(\frac{2}{3} \times \frac{3}{4}\right) = 0$

$$\Rightarrow \text{Cov}(X, Y) = \boxed{0}$$

c) $P(X > Y) = \int_0^1 \left(\int_y^1 2x \right) dx dy = \frac{2}{3} \Rightarrow P(X > Y) = \boxed{\frac{2}{3}}$

d) $f_{X,Y}(x, y) = \begin{cases} 6xy^2, & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0, & \text{otherwise} \end{cases}$

Joint PDF
of X and Y

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Question (3):

$$[a] P(190 < X < 200) = P\left(\frac{190 - 200}{15} < \overset{z}{\frac{X - 200}{15}} < \frac{200 - 200}{15}\right)$$

$$\Rightarrow P(190 < X < 200) = P\left(\frac{-2}{3} < z < 0\right) = P(-0.67 < z < 0)$$

$$\Rightarrow P(190 < X < 200) = \Phi(0) - \Phi(-0.67) = \Phi(0) - (1 - \Phi(0.67))$$

$$\Rightarrow P(190 < X < 200) = 0.5 - (1 - 0.7486) = 0.2486$$

$$[b] P(X > 230) = P\left(\overset{z}{\frac{X - 200}{15}} > \frac{230 - 200}{15}\right) = P(z > 2) = 1 - P(z < 2)$$

$$\Rightarrow P(X > 230) = 1 - 0.97725 = 0.02275$$

[c] S : denote the value such that 25% of the drinks are lesser than S

$$P(X < S) = 0.25 \Rightarrow P\left(\frac{X - 200}{15} < \frac{S - 200}{15}\right) = 0.25 \Rightarrow P\left(z < \frac{S - 200}{15}\right) = 0.25$$

$$\Rightarrow \text{From the table, we have } P(z < -0.67) = 0.25$$

$$\Rightarrow \frac{S - 200}{15} = -0.67 \Rightarrow S - 200 = -10.05 \Rightarrow S = 189.95$$

$$\Rightarrow \text{So, } S = 189.95 \Rightarrow \text{Below 189.95 milliliters we get the smallest 25\% of the drinks}$$

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Question (4):

Part (A):

$$\text{[1]} \int_0^1 f_x(x) dx = 1 \Rightarrow \int_0^{0.5} Ax dx + \int_{0.5}^1 (4-4x) dx = 1 \Rightarrow 0.125A + 0.5 = 1$$

$$A = \boxed{4}$$

$$\text{[2]} P(X \geq 0.55) = \int_{0.55}^1 f_x(x) dx = \int_{0.55}^{0.5} 4x dx + \int_{0.5}^1 (4-4x) dx$$

$$\Rightarrow P(X \geq 0.55) = \boxed{0.595}$$

Part (B):

[1]

	$Y=0$	$Y=1$	$Y=2$
$X=0$	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{1}{8}$
$X=1$	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{1}{8}$

$$\left. \begin{array}{l} \text{row 1} \\ \text{row 2} \end{array} \right\} \text{prob.} = \boxed{1} \checkmark$$

$$\text{[2]} \text{Cov}(X, Y) = E[XY] - E[X]E[Y] = \sum_{x,y} xy P(x, y) - E[X]E[Y]$$

$$\Rightarrow E[XY] = 0 \times 0 \times \frac{1}{8} + 0 \times 1 \times \frac{2}{8} + 0 \times 2 \times \frac{1}{8} + 1 \times 0 \times \frac{1}{8} + 1 \times 1 \times \frac{2}{8} + 1 \times 2 \times \frac{1}{8} = \frac{1}{2}$$

$$\Rightarrow E[X] = \frac{1}{2}, \quad E[Y] = 1$$

$$\Rightarrow \text{Cov}(X, Y) = \frac{1}{2} - \left(\frac{1}{2} \times 1\right) = 0$$

$$\Rightarrow \text{Cov}(X, Y) = \boxed{0}, \text{ So, } X \text{ and } Y \text{ are independent}$$

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Question (5):

Part (A): $n = 81$, $E[X_i] = \mu = 5$, $\text{Var}(X_i) = \sigma^2 = 4$
 $Y = X_1 + X_2 + \dots + X_{81}$

$$\begin{aligned}\Rightarrow P(Y > 369) &= P\left(\frac{Y - n\mu}{\sqrt{n}\sigma} > \frac{369 - n\mu}{\sqrt{n}\sigma}\right) \\ &= P\left(\frac{Y - 405}{18} > \frac{369 - 405}{18}\right) \\ &= P\left(\frac{Y - 405}{18} > -2\right)\end{aligned}$$

Approximately Standard Normal $\leftarrow$

$$\Rightarrow P(Y > 369) = 1 - \Phi(2) = 1 - 0.97725 = 0.2275$$

$$\Rightarrow P(Y > 369) = \boxed{0.2275}$$

Part (B):