

[1] $A - B = A \cap B^c$

[2] A and B are mutually exclusive if $A \cap B = \emptyset$ (Disjoint)

[3] collection of sets $A_1, A_2, A_3, \dots, A_n$ is a partition of S if:

$\Rightarrow$ They are disjoint

$\Rightarrow A_1 \cup A_2 \cup A_3 \cup \dots \cup A_n = S$

[4] $(A \cup B)^c = (\overline{A \cup B}) = A^c \cap B^c$ "De Morgan's Law"

[5] $\Rightarrow A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ "Distributive Law"

$\Rightarrow A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

[6] For any event A, $P(A) \geq 0$

[7] Probability of the Sample Space S, $P(S) = 1$

[8] For any countable collection $A_1, A_2, A_3, \dots$ of disjoint events:

$\Rightarrow P(A_1 \cup A_2 \cup A_3 \cup \dots) = P(A_1) + P(A_2) + P(A_3) + \dots$

[9] For mutually exclusive events A_1 and A_2 : $\Rightarrow$

$\Rightarrow P(A_1 \cup A_2) = P(A_1) + P(A_2)$

[10] If $A = (A_1 \cup A_2 \cup \dots \cup A_m)$ and $A_i \cap A_j = \emptyset$ for $i \neq j$: $P(A) = \sum_{i=1}^m P[A_i]$

[11] The probability of an event $B = \{s_1, s_2, \dots, s_m\}$ is: $P(B) = \sum_{i=1}^m P[\{s_i\}]$

[12] For an experiment with sample space $S = \{s_1, s_2, s_3, \dots, s_n\}$ in which each outcome s_i is equally likely:

$P(s_i) = \frac{1}{n}, \quad 1 \leq i \leq n$

[13] $P(A^c) = 1 - P(A)$

[14] $P(\emptyset) = P(S^c) = 0$

[15] $0 \leq P(A) \leq 1$

[16] $P(A - B) = P(A) - P(A \cap B)$

[17] $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

[18] $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$

$$[19] A \subset B \Rightarrow P(A) \leq P(B)$$

$$[20] P(A|B) = \frac{P(A \cap B)}{P(B)}, \text{ when } P(B) > 0 \quad \text{"Sample Space of } B \text{ is } B \text{ itself"}$$

$$[21] P(A|B) \geq 0$$

$$[22] P(B|B) = 1$$

$$[23] \text{ If } A_1, A_2, A_3, \dots \text{ are disjoint events, then: } \Rightarrow \text{"مجموع احتمالات"} \quad P(A_1 \cup A_2 \cup A_3 \cup \dots) = P(A_1) + P(A_2) + P(A_3) + \dots$$

$$\Rightarrow P(A_1 \cup A_2 \cup A_3 \cup \dots | B) = P(A_1|B) + P(A_2|B) + P(A_3|B) + \dots$$

$$[24] \text{ If } A \text{ and } B \text{ are disjoint } \Rightarrow P(A|B) = 0 = \frac{P(A \cap B)}{P(B)} = \frac{0}{P(B)}$$

$$[25] \text{ If } A \subset B \Rightarrow P(A|B) = \frac{P(A)}{P(B)} = \frac{P(A \cap B)}{P(B)}$$

$$[26] \text{ If } B \subset A \Rightarrow P(A|B) = 1 = \frac{P(A \cap B)}{P(B)} = \frac{P(B)}{P(B)}$$

$$[27] P(A) = \sum_{i=1}^m P(A_i) = \sum_{i=1}^m P(A \cap B_i) \quad \text{"Law of Total Probability"}$$

$$\Rightarrow P(A \cap B_i) = P(A_i)$$

$$[28] P(A|B_i) = \frac{P(A \cap B_i)}{P(B_i)} \Rightarrow P(A \cap B_i) = P(A|B_i) P(B_i) = P(A_i)$$

$$\Rightarrow \text{In general, } P(A) = \sum_{i=1}^m P(A_i) = \sum_{i=1}^m P(A \cap B_i) = \sum_{i=1}^m P(A|B_i) P(B_i) \quad \text{"Law of Total Probability"}$$

$$[29] P(B|A) = \frac{P(A|B) P(B)}{P(A)} \quad \text{"Bayes' Rule"}$$

$$[30] \text{ Two events } A \text{ and } B \text{ are independent if } A \text{ occurs without affecting the probability of } B \text{ occurring:}$$

$$P(A|B) = P(A), \text{ equivalently } P(A \cap B) = P(A) P(B)$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A) P(B)}{P(B)} = P(A)$$

[31] In Independency: $P(A|B) = P(A) \Rightarrow P(B|A) = P(B)$, $(P(A), P(B) \neq 0)$

[32] If A and B are independent, then:

$\Rightarrow$ A and B^c are independent

$\Rightarrow A^c$ and B are independent

$\Rightarrow A^c$ and B^c are independent

[32] In Independency: $P(A, \cup A_2) = 1 - (1 - P(A_1))(1 - P(A_2))$

[33] Three events A, B and C are ^{mutually} independent if all these conditions hold:

$$\left. \begin{aligned} \Rightarrow P(A \cap B) &= P(A)P(B) \\ \Rightarrow P(A \cap C) &= P(A)P(C) \\ \Rightarrow P(B \cap C) &= P(B)P(C) \\ \Rightarrow P(A \cap B \cap C) &= P(A)P(B)P(C) \end{aligned} \right\} \longrightarrow \begin{array}{l} \text{إذا شئت كل كلمتين مستقلتين} \\ \text{pair-wise independent} \end{array}$$

$\longrightarrow$ mutually independent : إذا شئت كل ثلاث كلمات

[1] For a finite sample space S with equally likely outcomes, the probability of an event A is given by:

$$P(A) = \frac{|A|}{|S|} = \frac{M}{N}$$

عدد النتائج الممكنة
في تجربة واحدة
الاحتمال

in order (الترتيب) Cardinality

[2] If we have r experiments, each experiment has n_i possible outcomes, then the total outcomes of the sequence of the r experiments = $n_1 \times n_2 \times n_3 \times \dots \times n_r$

[3] In the Ordered Sampling with Replacement (repetition allowed):

$\Rightarrow$ Number of outcomes = n^k
 عدد مرات التجربة
 Exp.
 عدد النتائج في Sample Space $\downarrow$

[4] In the Ordered Sampling without Replacement (repetition not allowed):

$\Rightarrow$ Number of outcomes = $n \times (n-1) \times (n-2) \times \dots \times (n-k+1)$

(عدد النتائج) nPr

$\Rightarrow$ Number of k -permutations of n distinguishable objects:

$$P_k^n = \frac{n!}{(n-k)!}, \text{ for } 0 \leq k \leq n$$

عدد النتائج الممكنة

$$P_k^n = n \cdot nPr \cdot k$$

باستخدام الـ
Permutations
(التباديل)

$\Rightarrow$ OR:

$$P_k^n = k! \cdot \binom{n}{k}$$

عدد النتائج الممكنة

$$P_k^n = k! \times (n \cdot nCr \cdot k)$$

باستخدام الـ
Combinations
(التوافيق)

$\{ \} \Rightarrow$ الترتيب غير مهم

$() \Rightarrow$ الترتيب مهم

5 In the Unordered Sampling without Replacement (Combination):

⇒ the number of k-combinations of n objects is :

n choose k →
$$\binom{n}{k} = \frac{P_k^n}{k!} = \frac{n!}{k!(n-k)!}$$

⇒ Number of ways to choose k objects out of n distinguishable objects = $\binom{n}{k}$

→
$$\binom{n}{k} = n {}^nC_k$$

6 If we have $n = n_1 + n_2 + \dots + n_r$, then the number of possible divisions of n distinct objects into r distinct groups of respective sizes $n_1, n_2, n_3, \dots, n_r$ is :

$$\binom{n}{n_1, n_2, n_3, \dots, n_r} = \frac{n!}{n_1! n_2! n_3! \dots n_r!}$$
 "Multinomial Coefficients"

مثال 30 : 140 شخصاً إلى 4 مجموعات بحجم 30، 40، 20، 50
 مثال 40 : 20 شخصاً إلى 2 مجموعات بحجم 10، 10



7 For n independent Bernoulli (Binomial) trials where each trial has success probability p, the probability of k success is given by :

$$P(k) = \binom{n}{k} p^k (1-p)^{n-k}$$
 ← outcome probability

⊕ Note:

$$\binom{N}{M} = \binom{8}{3} = \binom{8}{5} = 56$$

3+5=8
 3 و 5
 5 و 3

3

8 In the Unordered Sampling with Replacement (repetition allowed) :

⇒ the total number of distinct k samples from an n -element set such that repetition is allowed and ordering does not matter is :

$$\binom{n+k-1}{k}$$

9 The probability of n_0 failures and n_1 successes in $n = n_0 + n_1$ independent trials

is :

$$P[E_{\substack{\text{failure} \\ n_0}, \substack{\text{success} \\ n_1}}] = \binom{n}{n_1} (1-p)^{n-n_1} p^{n_1} = \binom{n}{n_0} (1-p)^{n_0} p^{n-n_0}$$

$\Downarrow$
using success

$\Downarrow$
using failure

① Conditions for Countable Set: $\rightarrow \boxed{N}, \boxed{Z}, \boxed{Q}$
 $\downarrow \quad \downarrow \quad \downarrow$
 Natural Nums. integers Rational Nums.

$\Rightarrow$ Finite set

$\Rightarrow$ One-to-one correspondence with Natural Numbers

$N = \{1, 2, 3, \dots\} \Rightarrow$ infinite set but countable (1) $\Rightarrow$ ∞

② If X is a discrete random variable, then:

$\Rightarrow \text{Range}(X) = S_X = \{x_1, x_2, x_3, \dots\}$

$\Rightarrow P_X(x_k) = P(X = x_k)$, for $k = 1, 2, 3, \dots$ (PMF)
 "Probability Mass Function of X "
 $\hookrightarrow$ Distribution

③ For a Discrete Random Variable with PMF $P_X(x)$ and Range $R_X = S_X = \{x_1, x_2, x_3, \dots\}$

$\Rightarrow 0 \leq P_X(x_k) \leq 1$, for all $x_k \in S_X$

$\Rightarrow \sum_{k=1}^{\infty} P_X(x_k) = 1$

$\Rightarrow B \subset S_X, P(X \in B) = P(B) = \sum_{x_k \in B} \text{Prob}\{X = x_k\} = \sum_{x_k \in B} P_X(x_k)$

$\rightarrow$ outputs ② $\hookrightarrow$

④ Bernoulli Random Variables: $X \sim \text{Bernoulli}(p) \rightarrow$ $\hookrightarrow$ نجاح success

$\Rightarrow \text{PMF} = P_X(x) = \begin{cases} p & , \text{ for } x=1 \Rightarrow \text{success} \\ 1-p & , \text{ for } x=0 \Rightarrow \text{fail} \\ 0 & , \text{ otherwise} \end{cases}$

$\Rightarrow \text{Range}(X) = \{0, 1\}$, $P_X(0) = 1-p$, $P_X(1) = p$
 $\downarrow \quad \downarrow$
 fail $\hookrightarrow$ ② $\downarrow$ success $\hookrightarrow$ ①

2

كل د. ب. ب. في
Bernoulli

5 Geometric Random Variables : $X \sim \text{Geometric}(p)$

probability of success

بفضل أي د. ب. ب. في
(p) success

$$\Rightarrow \text{PMF} = P_X(k) = \begin{cases} p(1-p)^{k-1} & , k = 1, 2, 3, \dots \\ 0 & , \text{otherwise} \end{cases}$$

$$\Rightarrow S_X = \text{Range}(X) = \{1, 2, 3, \dots\} = \mathbb{N}$$

times [n] د. ب. ب. في
observation

6 Binomial Random Variables : $X \sim \text{Binomial}(n, p)$

Bernoulli

success

$$\Rightarrow \text{PMF} = P_X(k) = \begin{cases} \binom{n}{k} p^k (1-p)^{n-k} & , k = 0, 1, 2, \dots, n \\ 0 & , \text{otherwise} \end{cases}$$

$$\Rightarrow S_X = \text{Range}(X) = \{0, 1, 2, \dots, n\}$$

$$\Rightarrow n \times p = \text{the value with highest PMF} \quad (\text{التركيز هنا، هنا})$$

3

هو كامة بردي أعيد ال Bernoulli عشوائياً
ألاحظ ال H مثلاً m مرة

7 **Pascal Random Variable** : $X \sim \text{Pascal}(m, p)$

ال Geometric Distribution هو ال Pascal Distribution ال
" m = 1 بكونه "

$$0 < p < 1$$

(ال probability ال success)

$$\Rightarrow \text{PMF} = P_x(k) = \begin{cases} \binom{k-1}{m-1} p^m (1-p)^{k-m}, & k = m, m+1, m+2, \dots \\ 0, & \text{otherwise} \end{cases}$$

$$\Rightarrow S_x = \text{Range}(X) = \{m, m+1, m+2, \dots\}$$

ال Geometric Distribution بقينا نرى نقطة التقاء ال H مرة واحدة
ما بظل نضيف ال H نطلب m من المرات

8 **Uniform Random Variable** : $X \sim \text{Uniform}(S_x)$

ال R_x

$$\Rightarrow P_x(x_i) = \frac{1}{|S_x|}$$

ال Cardinality ال Range

$$\Rightarrow S_x = \text{Range}(X) = \{x_1, x_2, x_3, \dots\}$$

ال مثلاً : رمينا حجر نرد : $X = \{1, 2, 3, 4, 5, 6\}$
 $\left(\frac{1}{6}\right) \left(\frac{1}{6}\right) \left(\frac{1}{6}\right) \left(\frac{1}{6}\right) \left(\frac{1}{6}\right) \left(\frac{1}{6}\right)$

Q1 **Poisson Random Variable** : $X \sim \text{Poisson}(\alpha)$ $\rightarrow$ E.x. the average number of arrivals in that interval

$$\alpha = \lambda T$$

$\Rightarrow$ It is used to model :

- * Arrival of customers at a service facility
- * Arrival of information request at a server

In General, Counting the occurrence of certain events in an interval of time or space

$$\Rightarrow \text{PMF} = P_X(k) = \begin{cases} \frac{e^{-\alpha} \alpha^k}{k!}, & k = 0, 1, 2, 3, \dots \\ 0, & \text{otherwise} \end{cases}$$

$$\Rightarrow S_X = \text{Range}(X) = \{0, 1, 2, 3, \dots\}$$

$$\Rightarrow \text{For Examples } \alpha = \lambda T = (2) \text{ hits/s} \times (0.25) \text{ s} = 0.5$$

5

10 If X is any random variable, the Cumulative Distribution Function (CDF) of X is :

$$F_X(x) = P(X \leq x), \text{ for all } x \in \mathbb{R}$$

11 If X is a discrete random variable with range $S_X = \{x_1, x_2, x_3, \dots\}$:

$$\Rightarrow F_X(-\infty) = P(X < -\infty) = 0$$

$$\Rightarrow F_X(+\infty) = 1$$

$$\Rightarrow \text{If } y \geq x \Rightarrow F_X(y) \geq F_X(x)$$

$$\Rightarrow x_i \in S_X, F_X(x_i) - F_X(x_i - \epsilon) = P_X(x_i), \text{ for } \epsilon > 0 \text{ small enough}$$

$\Rightarrow$ For a discrete random variable X , there is a jump (discontinuity) at each value of $x_i \in S_X$. The height of the jump at x_i is $P_X(x_i) \Rightarrow$ PMF

$$\Rightarrow x_i \leq x < x_{i+1} \Rightarrow F_X(x) = F_X(x_i)$$

$$CDF(b) - CDF(a)$$

$$\Rightarrow \text{For all } a \leq b, \text{ we have: } P(a < X \leq b) = F_X(b) - F_X(a) \Rightarrow$$

عالم گانه
احتمال
probability
range

12 If X is a discrete random variable with range $S_X = \{x_1, x_2, x_3, \dots\}$, then :

$$\Rightarrow E[X] = \mu_X = \sum_{x_k \in S_X} x_k P(X = x_k) = \sum_{x_k \in S_X} x_k P_X(x_k)$$

"Expected Value"
"Mean Value"
"Average"

$$\Rightarrow P_X(x_{\text{mod}}) \geq P_X(x) \text{ for all } x$$

"Mode of random variable X "

احتمال بزرگتر و بیشتر از بقیه

range of x

$$\Rightarrow P[X \leq x_{\text{med}}] \geq \frac{1}{2}, P[X \geq x_{\text{med}}] \geq \frac{1}{2}$$

"Median of random variable X "

نصف از بقیه احتمال بزرگتر از نصف

احتمال بزرگتر از نصف

[13] For $X \sim \text{Bernoulli}(p)$:

$$\Rightarrow E[X] = p$$

[14] For $X \sim \text{Geometric}(p)$:

$$\Rightarrow E[X] = \frac{1}{p}$$

[15] For $X \sim \text{Poisson}(\alpha)$:

$$\Rightarrow E[X] = \alpha$$

[16] For $X \sim \text{Binomial}(n, p)$:

$$\Rightarrow E[X] = np$$

[17] For $X \sim \text{Pascal}(k, p)$:

$$\Rightarrow E[X] = k/p$$

[18] For $X \sim \text{Uniform}(k, l)$:

$$\Rightarrow E[X] = \frac{(k+l)}{2}$$

[19] If X is a random variable and $Y = g(X)$, then Y itself is a random variable:

$$\Rightarrow S_X = \{x_1, x_2, x_3, \dots\}$$

$$\Rightarrow S_Y = \{g(x) \mid x \in S_X\} = \{g(x_1), g(x_2), \dots\}$$

$$\Rightarrow P_Y(y_k) = P(Y = y_k) = P(g(x) = y_k)$$

[20] Law of the unconscious statistician (LOTUS) for discrete random variables:

$$\Rightarrow E[g(X)] = \mu_Y = \sum_{x_k \in S_X} \overset{\text{(value)}}{g(x_k)} \underset{\substack{\text{each} \\ \text{range } x}}{P_X(x_k)} \quad \text{"Expected Value", "Mean", "Average"}$$

[21] Linearity of expectation:

$$\Rightarrow Y = aX + b \Rightarrow E[Y] = aE[X] + b, \quad a, b \in \mathbb{R}$$

[22] The **Variance** of a random variable X with mean $E[X] = \mu_X$ is defined as:

$$\text{Var}(X) = E[X^2] - \underbrace{(E[X])^2}_{\mu_X^2}$$

[23] The **Standard Deviation** of a random variable X is defined as:

$$\sigma_X = \sqrt{\text{Var}(X)}$$

[24] For a random variable X and real numbers a and b :

$$\text{Var}(aX + b) = a^2 \text{Var}(X) \Rightarrow \text{"mean shifts by } a \text{ to } a\mu + b"$$

$$E[aX + b] = aE[X] + b$$

[25] If $X_1, X_2, \dots, X_n$ are independent random variables and $X = X_1 + X_2 + \dots + X_n$, then:

$$\text{Var}(X) = \text{Var}(X_1) + \text{Var}(X_2) + \dots + \text{Var}(X_n)$$

[26] For a random variable X :

$\Rightarrow$ The n th moment is: $E[X^n]$

$\Rightarrow$ The n th central moment is: $E[(X - \mu_X)^n]$

$\xrightarrow{\text{new } n=2 \text{ is variance}}$

① For a Continuous Random Variable X with an absolutely continuous CDF :

$$\Rightarrow f_x(x) = \frac{dF_x(x)}{dx} = F'_x(x) = \text{"Probability Density Function" (PDF)}$$

$$\Rightarrow F_x(x) = \int_{-\infty}^x f_x(\alpha) d\alpha = \text{"Cumulative Distribution Function" (CDF)}$$

$$\textcircled{2} P(a < X < b) = F_x(b) - F_x(a) = \int_a^b f_x(\alpha) d\alpha \Rightarrow \begin{array}{l} \text{CDF} \downarrow \text{استخدام} \\ \text{PDF} \downarrow \text{استخدام} \end{array} \quad \begin{array}{l} \text{interval} \downarrow \text{probability} \end{array}$$

$$\textcircled{3} \int_{-\infty}^{\infty} f_x(x) dx = 1 \Rightarrow \text{Sample Space} \downarrow \text{Summation} \downarrow \text{Discrete} \downarrow \text{is} , \sum(S) = 1$$

④ Since $F_x(x)$ is monotone non-decreasing, its derivative must satisfy :

$$f_x(x) \geq 0 , \text{ for all } x \in \mathbb{R}$$

$$\textcircled{5} \int_{-\infty}^{\infty} f_x(u) du = F_x(\infty) - F_x(-\infty) = 1 - 0 = 1$$

⑥ The Expected Value of a Continuous Random Variable X is :

$$\overset{\text{first moment} \leftarrow}{E[X]} = \int_{-\infty}^{\infty} x f_x(x) dx$$

⑦ Law of the unconscious statistician (LOTUS) for Continuous Random Variable :

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_x(x) dx$$

↓
function of
Random Variable

2

8 For a Continuous Random Variable, the Variance is:

$$\text{Var}(X) = E[X^2] - (E[X])^2 = \int_{-\infty}^{\infty} x^2 f_x(x) dx - (\mu_x)^2$$

(second moment)

9 For a Continuous Random Variable, the Standard Deviation is:

$$\sigma = \sqrt{\text{Var}(X)}$$

10 $X \sim \text{Uniform}(a, b)$:

$$\Rightarrow \text{PDF} = f_x(x) = \begin{cases} \frac{1}{b-a}, & a < x < b \\ 0, & x < a \text{ or } x > b \end{cases}$$

$$\Rightarrow \text{CDF} = F_x(x) = \begin{cases} 0, & \text{for } x < a \\ \frac{x-a}{b-a}, & \text{for } a \leq x \leq b \\ 1, & \text{for } x > b \end{cases}$$

$$\Rightarrow E[X] = \frac{a+b}{2}$$

$$\Rightarrow \text{Var}(X) = \frac{(b-a)^2}{12} \Rightarrow a < b$$

3

III $X \sim \text{Exponential}(\lambda)$:

في شبكة اتصالات، كل packet
يحتاج وقتاً معيناً ليتم إرساله
Network time delay

وقت إنتظار في شبكة اتصالات
Waiting time in a network

$$\Rightarrow \text{PDF} = f_x(x) = \begin{cases} \lambda e^{-\lambda x} & , x \geq 0, \lambda > 0 \\ 0 & , \text{else} \end{cases} \Rightarrow \lambda e^{-\lambda x} u(x)$$

$$\Rightarrow \text{CDF} = F_x(x) = \begin{cases} (1 - e^{-\lambda x}) & , x \geq 0 \\ 0 & , \text{otherwise} \end{cases} \Rightarrow (1 - e^{-\lambda x}) u(x)$$

$$\Rightarrow E[X] = \frac{1}{\lambda}$$

"Expected Value", "Mean Value", "Average"

$$\Rightarrow \text{Var}(X) = \frac{1}{\lambda^2}$$

"Variance"

12 $X \sim \text{Erlang}(n, \lambda)$:

n exponential random variables

وقت إنتظار في شبكة اتصالات
Waiting time in a network

$$\Rightarrow \text{PDF} = f_x(x) = \begin{cases} \frac{\lambda^n x^{n-1} e^{-\lambda x}}{(n-1)!} & , x \geq 0, n \geq 1, \lambda > 0 \\ 0 & , \text{otherwise} \end{cases}$$

$$\Rightarrow E[X] = \frac{n}{\lambda}$$

"Expected Value", "Mean Value", "Average"

$$\Rightarrow \text{Var}(X) = \frac{n}{\lambda^2}$$

"Variance"

4

13 $X \sim \text{Gaussian}(\overset{E[X]}{\mu}, \overset{\text{Standard Deviation}}{\sigma})$: $\xrightarrow{\text{alt.}}$

- ① توزيع درجات الطلاب في سابقه
- ② تغير درجات الحرارة في شهر معين
- ③ noise في ال systems
- ④ error في communication system

$\Rightarrow \text{PDF} = f_X(x) = \begin{cases} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} & , \sigma > 0, \mu \text{ any real number} \\ 0 & \text{otherwise} \end{cases}$

(peek)

14 If $X \sim \text{Gaussian}(\mu_X, \sigma_X)$, and $Y = aX + b$, where $a, b \in \mathbb{R}$, then:

$\Rightarrow Y \sim \text{Gaussian}(\mu_Y, \sigma_Y)$

$\Rightarrow E[Y] = a E[X] + b = a\mu_X + b$

$\Rightarrow \text{Var}(Y) = a^2 \text{Var}(X) = a^2 \sigma_X^2$

15 $Z \sim \text{Standard Normal Distribution}(\overset{\mu}{0}, \overset{\sigma}{1})$: $\Rightarrow \boxed{\text{Gaussian Distribution}}$

$Z \sim N(0, 1)$

$\Rightarrow \text{PDF} = f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$, for all $z \in \mathbb{R}$

$\Rightarrow \text{CDF} = \Phi(x) = P(X \leq x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{u^2}{2}} du \Rightarrow \text{Only when } \mu=0, \sigma=1$

5

$$X \sim N(\mu, \sigma)$$

[16] If X is a Gaussian (μ, σ) random variable, the CDF of X is:

$$\Rightarrow \text{CDF} = F_X(x) = \Phi\left(\frac{x - \mu}{\sigma}\right)$$

$\Rightarrow$ The probability that X is in the interval $(a, b]$ is:

$$P[a \leq X \leq b] = F_X(b) - F_X(a) = \Phi\left(\frac{b - \mu}{\sigma}\right) - \Phi\left(\frac{a - \mu}{\sigma}\right)$$

[17] Properties of the Φ function:

$$\Rightarrow \lim_{x \rightarrow \infty} \Phi(x) = 1, \quad \lim_{x \rightarrow -\infty} \Phi(x) = 0$$

$$\Rightarrow \Phi(0) = \frac{1}{2}$$

$$\Rightarrow \Phi(-x) = 1 - \Phi(x), \quad \text{for all } x \in \mathbb{R}$$

$$\Rightarrow \Phi(-x) + \Phi(x) = 1$$

[18] If X is Standard Normal Random Variable, $X \sim N(\mu, \sigma)$, then:

$$\Rightarrow P(|X - \mu| \leq 1\sigma) = P(\mu - \sigma \leq X \leq \mu + \sigma) = \Phi\left(\frac{\mu + \sigma - \mu}{\sigma}\right) - \Phi\left(\frac{\mu - \sigma - \mu}{\sigma}\right)$$

$\Downarrow$
 First Standard Deviation

$\Phi(1) - \Phi(-1)$

$$\Rightarrow P(|X - \mu| \leq 3\sigma) = P(\mu - 3\sigma \leq X \leq \mu + 3\sigma) = \Phi\left(\frac{\mu + 3\sigma - \mu}{\sigma}\right) - \Phi\left(\frac{\mu - 3\sigma - \mu}{\sigma}\right)$$

$\Downarrow$
 Third Standard Deviation

$\Phi(3) - \Phi(-3)$

6

19 The Standard Normal Complementary CDF is: $\boxed{\text{CCDF}} = 1 - \text{CDF} = 1 - \Phi(z)$

$$Q(z) = P[Z > z] = \frac{1}{\sqrt{2\pi}} \int_z^{\infty} e^{-u^2/2} du = 1 - \Phi(z)$$

$\left(\frac{z-\mu}{\sigma}\right) \Rightarrow$ standard (with 1st Gaussian) p.d.f.

20 If the behaviour of a continuous random variable X is described by the distribution $N(\mu, \sigma^2)$, then the behaviour of the random variable $Z = \frac{X-\mu}{\sigma}$ is described by the standard normal distribution $N(0, 1)$

$\Rightarrow$ We call Z the Standardised Normal Variable $\Rightarrow Z \sim N(0, 1)$

$$0.1 = \Phi(x) \Rightarrow x = 1.28$$

$$1 - \Phi(x) = 0.1 \Rightarrow x = 1.28$$

$$\frac{1}{2} = \Phi(0) \Rightarrow$$

$$\Phi(x) + \Phi(-x) = 1 \Rightarrow \Phi(x) = 1 - \Phi(-x)$$

$$1 = \Phi(x) + \Phi(-x) \Rightarrow$$

with $(\mu, \sigma^2) \sim X$, standard normal random variable $Z \sim N(0, 1)$

$$\left(\frac{\mu - \sigma z}{\sigma}\right) \Phi - \left(\frac{\mu + \sigma z}{\sigma}\right) \Phi = (\mu - \sigma z \geq X \geq \mu + \sigma z) \Phi = (\sigma \geq |\mu - X|) \Phi \Rightarrow$$

$$\left(\frac{\mu - \sigma z}{\sigma}\right) \Phi - \left(\frac{\mu + \sigma z}{\sigma}\right) \Phi = (\mu - \sigma z \geq X \geq \mu + \sigma z) \Phi = (\sigma \geq |\mu - X|) \Phi \Rightarrow$$

① Joint Probability Mass Function (PMF) for X and Y :

$$\Rightarrow P_{x,y}(x_k, y_j) = P(X = x_k \text{ and } Y = y_j)$$

(pairs) (and) (pairs)

Range $\Rightarrow R_{xy} = \text{all possible value for } (X, Y) = \{(x_i, y_j) \mid x_i \in R_x, y_j \in R_y\}$

$$\Rightarrow \sum_{(x_i, y_j) \in R_{xy}} P_{xy}(x_i, y_j) = 1$$

② For discrete random variables X and Y and any set B in the (X, Y) plane, the probability of the event $\{(X, Y) \in B\}$ is:

$$P[B] = \sum_{(x,y) \in B} P_{x,y}(x,y)$$

③ For discrete random variables X and Y with joint PMF $P_{x,y}(x,y)$:

$$\Rightarrow P_x(x) = \sum_{y \in R_y} P_{x,y}(x,y)$$

probability of x if $x \in R_x$
 $Y \in \text{Range of } Y$

$$\Rightarrow P_y(y) = \sum_{x \in R_x} P_{x,y}(x,y)$$

probability of y if $y \in R_y$
 $X \in \text{Range of } X$

"Marginal PMF"

④ The Joint Cumulative Distribution Function of two random variables:

$$F_{xy}(x,y) = P(X \leq x \text{ and } Y \leq y)$$

(and) (pairs)

2

5 Properties of Joint CDF: X and Y are random variables.

$$\Rightarrow F_{xy}(x, \infty) = P(X \leq x, Y < \infty) = P(X \leq x) = F_x(x)$$

$$\Rightarrow F_{xy}(\infty, y) = P(X < \infty, Y < y) = P(Y \leq y) = F_y(y)$$

$$\Rightarrow F_{xy}(\infty, \infty) = 1$$

$$\Rightarrow F_{xy}(-\infty, y) = 0, \text{ for any } y$$

$$\Rightarrow F_{xy}(x, -\infty) = 0, \text{ for any } x$$

$$\Rightarrow 0 \leq F_{xy}(x, y) \leq 1$$

6 For two random variables X and Y , and real numbers $x_1 \leq x_2$, $y_1 \leq y_2$, we have:

$$P(x_1 < X < x_2, y_1 < Y < y_2) = F_{xy}(x_2, y_2) - F_{xy}(x_1, y_2) - F_{xy}(x_2, y_1) + F_{xy}(x_1, y_1)$$

3

[7] Two random variables X and Y are jointly continuous if there exists a nonnegative function $f_{xy} : \mathbb{R}^2 \rightarrow \mathbb{R}$, such that, for any set $A \in \mathbb{R}^2$, we have:

$$P((X, Y) \in A) = \iint_A f_{xy}(x, y) dx dy \quad \text{"Joint Probability Density Function"}$$

Joint PDF

[8] The Joint CDF is:

$$F_{xy}(x, y) = P(X \leq x, Y \leq y) = \int_{-\infty}^y \int_{-\infty}^x f_{xy}(u, v) du dv$$

[9] Properties of Joint CDF:

$$\Rightarrow F_{xy}(x, \infty) = F_x(x), \text{ for any } x \quad \text{"marginal CDF of } X \text{"}$$

$$\Rightarrow F_{xy}(\infty, y) = F_y(y), \text{ for any } y \quad \text{"marginal CDF of } Y \text{"}$$

$$\Rightarrow F_{xy}(\infty, \infty) = 1$$

$$\Rightarrow F_{xy}(-\infty, y) = F_{xy}(x, -\infty) = 0$$

$$\Rightarrow P(x_1 < X \leq x_2, y_1 < Y \leq y_2) = F_{xy}(x_2, y_2) - F_{xy}(x_1, y_2) - F_{xy}(x_2, y_1) + F_{xy}(x_1, y_1)$$

[10] For Continuous Random Variables, the Marginal PDF is:

$$\Rightarrow f_x(x) = \int_{-\infty}^{\infty} f_{xy}(x, y) dy, \text{ for all } x \quad \text{"Marginal PDF of } X \text{"}$$

$$\Rightarrow f_y(y) = \int_{-\infty}^{\infty} f_{xy}(x, y) dx, \text{ for all } y \quad \text{"Marginal PDF of } Y \text{"}$$

ii) Two Random Variables X and Y are independent if: X and Y are independent $\Leftrightarrow T/F$

١) PMF $\Leftrightarrow$ * $P_{xy}(x_i, y_j) = \underline{P_x(x_i)} \underline{P_y(y_j)}$, for all $x_i, y_j \Rightarrow$
 PDF $\Uparrow$ ٢) (مربوع المستقلات)
 * $F_{xy}(x, y) = F_x(x) F_y(y)$, for all x, y

We can use
marginal PDFs
PMFs

12 Let X and Y are two random variables, and suppose that $Z = g(X, Y)$, Z is random variable, and $\therefore$ Function of X and Y

$$\Rightarrow E[g(X, Y)] = \sum_{(x_i, y_j) \in R_{XY}} g(x_i, y_j) \underbrace{P_{XY}(x_i, y_j)}_{\text{PMF}} \quad \text{"Expected Value"} \quad \text{Discrete}$$

$$\Rightarrow E[g(x, y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) \underbrace{f_{xy}(x, y)}_{\text{PDF}} dx dy \quad \text{"Expected Value"} \quad \text{Continuous}$$

Linear Operation $\Rightarrow E[a_1 g_1(x, y) + \dots + a_n g_n(x, y)] = a_1 E[g_1(x, y)] + \dots + a_n E[g_n(x, y)]$

[13] For any two random variables X and Y :

$$E[X+Y] = E[X] + E[Y] \Rightarrow \text{Linearity}$$

14) The Variance of the sum of two random variables is:

$$\text{Var}[X+Y] = \text{Var}[X] + \text{Var}[Y] + 2E[(X-\underbrace{\mu_X}_{E[X]})(Y-\underbrace{\mu_Y}_{E[Y]})]$$

5

عوامل عشوائية
② Random Variables

15 The **Covariance** between X and Y is defined as: $\Rightarrow$ لو بين $\oplus$ يعني العلاقة بين متغيرات
لو بين $\ominus$ يعني العلاقة بين متغيرات

$$\text{Cov}(X, Y) = E[(X - E[X])(Y - E[Y])] = E[XY] - (E[X])(E[Y])$$

16 Properties of Covariance:

$$\Rightarrow \text{Cov}(X, X) = \text{Var}(X)$$

$$\Rightarrow \text{If } X \text{ \& } Y \text{ independent, then } \text{Cov}(X, Y) = 0$$

$$\Rightarrow \text{Cov}(X, Y) = \text{Cov}(Y, X)$$

$$\Rightarrow \text{Cov}(aX, Y) = a \text{Cov}(X, Y), \quad a \in \mathbb{R} \quad \Leftarrow \text{الضرب يؤثر على Cov}$$

$$\Rightarrow \text{Cov}(X+c, Y) = \text{Cov}(X, Y) \quad \Leftarrow \text{لكن المبع لا يؤثر على Cov}$$

$$\Rightarrow \text{Cov}(X+Y, Z) = \text{Cov}(X, Z) + \text{Cov}(Y, Z)$$

$$\Rightarrow \text{Cov}(X+Y, Z+W) = \text{Cov}(X, Z) + \text{Cov}(X, W) + \text{Cov}(Y, Z) + \text{Cov}(Y, W)$$

(علاقہ صحت و التواضع)

17 The Correlation Coefficient is:

$$\rho_{xy} = \rho(X, Y) = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X) \text{Var}(Y)}} = \frac{\text{Cov}(X, Y)}{\sigma_x \sigma_y}$$

[18] The properties of Correlation Coefficient :

$\Rightarrow -1 \leq \rho(x, y) \leq 1$

$\Rightarrow$ variables independent if $P(X, Y) = 0$ not like

$\Rightarrow \rho(ax+b, cy+d) = \rho(x, y)$, for $a, c > 0 \Rightarrow$ الجمع والاضرب لا يغيرون ρ الى ρ

$$\Rightarrow \rho(x, y) = 1 \text{ if } y = ax + b, \quad a > 0$$

$$\Rightarrow \rho(x, y) = \boxed{-1} \text{ if } y = ax + b, \quad a < 0$$

[19] Consider two random variables X and Y : $(S, X), (U, S, Y, X), (Z, Y, X) \in$

$\Rightarrow$ If $\rho(X, Y) = 0 \longrightarrow X$ and Y are **uncorrelated**

⇒ If $\rho(x, y) > 0 \longrightarrow X$ and Y are positively correlated

$\Rightarrow$ If $\rho(x, y) < 0 \longrightarrow X$ and Y are **negatively correlated**

7

cross-correlation

[20] The Correlation of X and Y is $r_{x,y} = E[XY]$ $\Rightarrow$ یعنی میانگین حاصل ضرب

[21] Random Variables X and Y are ^{معمولی} orthogonal if $r_{x,y} = 0$

[22] Random Variables X and Y are uncorrelated if $Cov[X, Y] = 0$

[23] For independent random variables X and Y :

$$\Rightarrow E[g(x)h(y)] = E[g(x)]E[h(y)]$$

$$\Rightarrow r_{x,y} = E[XY] = E[X]E[Y]$$

$$\Rightarrow Cov[X, Y] = \rho_{x,y} = 0$$

$$\Rightarrow Var[X+Y] = Var[X] + Var[Y] = 0$$