

Chapter 4 $\Rightarrow$ Time Response.

Time Response = Transient Response + Steady state R_{ss}

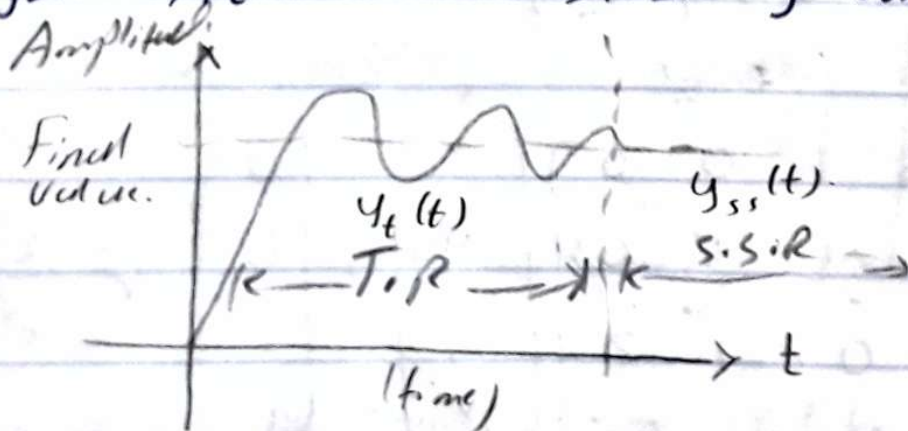
- Transient Response $\rightarrow$ "Natural Response".

is the part of time response that goes to zero at $t \rightarrow \infty$

- Steady state Response $\rightarrow$ "Forced - Response"

is the part of time response that remains after the transient has died out. (totally depending on input).

HP² Note $\Rightarrow$ To Find the time response for the system, we need to check the stability of this system, it must be stable system.

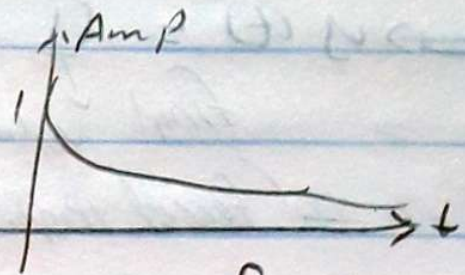


Example 1: Find the step-time Response for system.

$$G(s) = \frac{s}{s+2}, \quad R(s) = \frac{1}{s}$$

$$G(s) = \frac{Y(s)}{R(s)} = \frac{s}{s+2} \Rightarrow Y(s) = \frac{1}{s+2}$$

$$\Rightarrow y(t) = \underbrace{e^{-2t}}_{\text{transient response}} u(t)$$



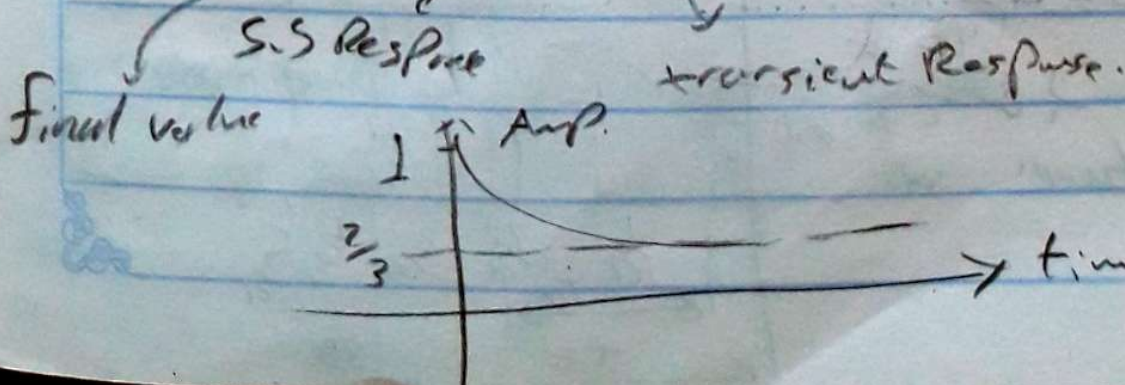
NO forced response this means no final value when $t \rightarrow \infty$. when $t \rightarrow \infty \Rightarrow y(t) = 0$.

Example 2: $G(s) = \frac{s+2}{s+3}$, Find step response $R(s) = \frac{1}{s}$

$$Y(s) = \frac{s+2}{s+3} \Rightarrow Y(s) = \frac{s+2}{s(s+3)} = \frac{K_1}{s} + \frac{K_2}{s+3}$$

$$K_1 = \frac{2}{3}, \quad K_2 = -\frac{1}{3}$$

$$\Rightarrow y(t) = \left(\frac{2}{3} + \frac{1}{3} e^{-3t} \right) u(t)$$



* First order system is .
 - general form:- $G(s) = \frac{a}{s+a}$

If $R(s) = \frac{1}{s}$, the step response will be equal

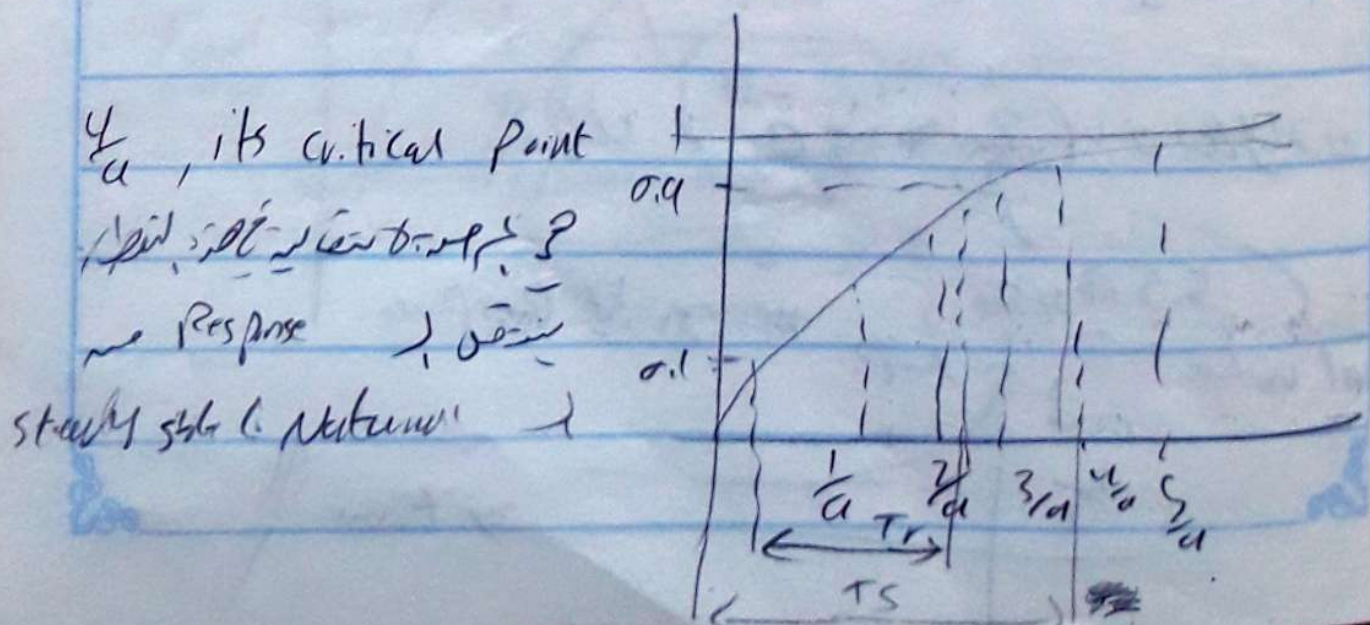
$$\Rightarrow y(t) = \underbrace{(1)}_{\text{final value}} - \underbrace{e^{-at}}_{\text{natural response}} u(t)$$

(forced response)

$$G(s) = \frac{\text{DC-gain}}{s + \frac{1}{\text{time Constant (T)}}}$$

- Time Constant :-

It is the time required to reach 63% of the steady state value or final value for step input or to decrease to 37% of the initial value



- settling time (T_s) :-

Is the time required to reach to 2% from the final value and it is equal.

$$T_s \approx \frac{4}{a}$$

- Rise time (T_r) :-

is the time required to go from 10% to 90% from the final value and it is equal

$$T_r \approx \frac{2.2}{a}$$

Notes :- From this relations we can found if the value of (a) is larger that means the system is more faster than the system has small value of (a).

* Example: $G(s) = \frac{10}{s+5}$, Find the following

DC-gain, T_c , T_s , T_r .

$$G(s) = \frac{10/5}{\frac{1}{5}s+1} \quad \text{DC-gain}$$

$$\text{DC gain} = 2, \quad T_s = \frac{4}{a} = \frac{4}{5} \text{ sec}$$

$$T_c = \frac{1}{5} \text{ sec}, \quad T_r \approx \frac{2.2}{5} \text{ sec}$$

* Second-order systems :-

General form :-

$$G(s) = \frac{a}{s^2 + bs + c} = \frac{K_{dc} \omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

- K_{dc} is the DC gain $[G(s)]_{s=0}$ or final value input value
it's the main reason for steady state error.

- ζ is the damping ratio.

- ω_n is the natural frequency.

* 4 cases of response.

(1) overdamped (2) underdamped

(3) Critically damped (4) underdamped.

These type step time response depend poles locations.

(1) Overdamped Response:- (Poles:- Real & distinct)

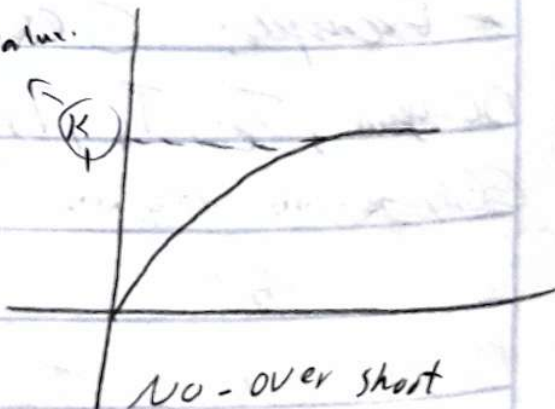
$$G(s) = \frac{s+c}{(s+a)(s+b)}$$

$$Y(s) = \frac{K_1}{s} + \frac{K_2}{s+a} + \frac{K_3}{s+b}$$

Forced
Response.

Natural Response

final value.



No-overshoot

All poles in LHP.

The system is stable

12) Under damped Response (poles Complex).

$$G(s) = \frac{1}{s + c}$$

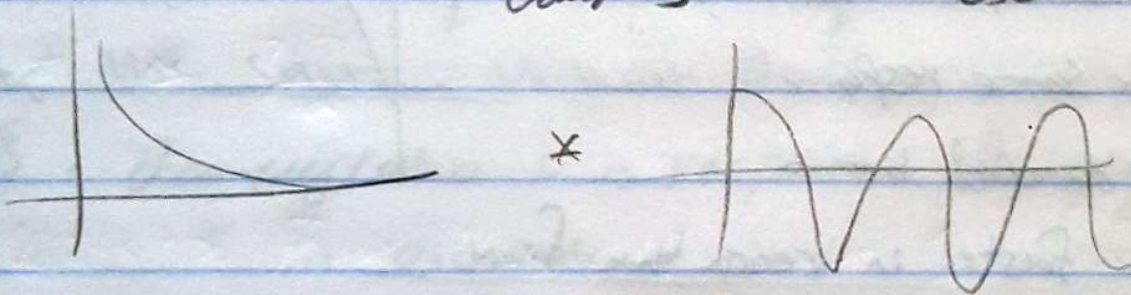
From input

$$Y(s) = \frac{K_1}{s} + \frac{K_2}{s + \alpha - j\beta} + \frac{K_2^*}{s + \alpha + j\beta}$$

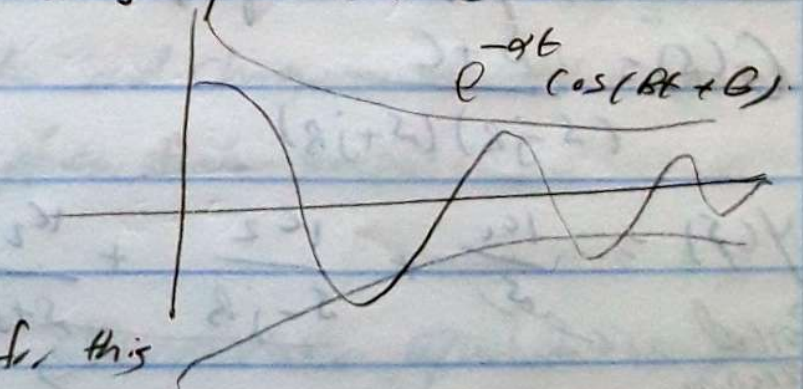
forced Response

Natural Response.

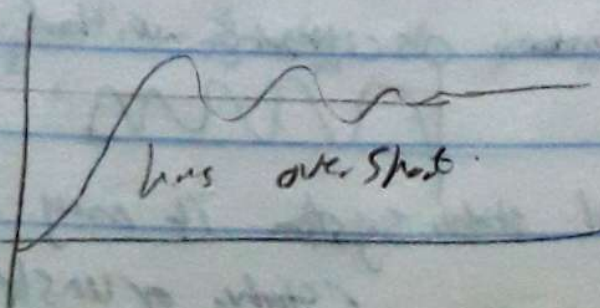
In the time domain $\Rightarrow 2|K_2| e^{-\alpha t} \cos(\beta t + \theta^\circ)$
 damping oscillation.



So, the real part of the complex poles is responsible for damping and the Imag. part for oscillation.



So, the time Response for this type is shown.



③ Critically damped Response (Rise and Response)

$$G(s) = \frac{s+c}{(s+c)^2}$$

$$Y(s) = \frac{K_1}{s} + \frac{K_2}{(s+c)^2} + \frac{K_3}{s+c}$$

forced
Response

Natural Response

- No overshoot



- gives the same response

of overdamped but it

is more faster in reaching the final

④ Undamped response :- (Imaginary poles)

$$G(s) = \frac{s+c}{(s-j\beta)(s+j\beta)}$$

$$Y(s) = \frac{K_1}{s} + \frac{K_2}{s-j\beta} + \frac{K_3}{s+j\beta}$$

forced
Response

Natural Response

$$y(t) = K_1 + 2(K_2) \cos(\beta t + \phi)$$

continuous oscillation without decay

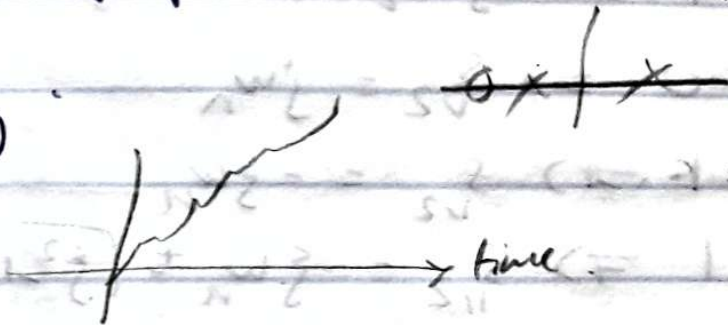
there is no final value.



This system called marginal stable system it may be (stable or unstable)

Notes: Unstable response (Poles Positive in RHP)

$$G(s) = \frac{s+c}{(s-a)(s+b)}$$



* The general second order system: -

ω_n : natural freq = is the freq oscillation of the system without damping.

ξ : damping ratio is responsible for damping in oscillation

$\Rightarrow$ So the value of these quantities change in the characteristic equation and change in the pole location and change in the type of the time response.

- The formula of characteristic equation is:

$$s^2 + 2\xi\omega_n s + \omega_n^2 = 0.$$

$$s_{1,2} = -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}$$

So if the ξ equal:

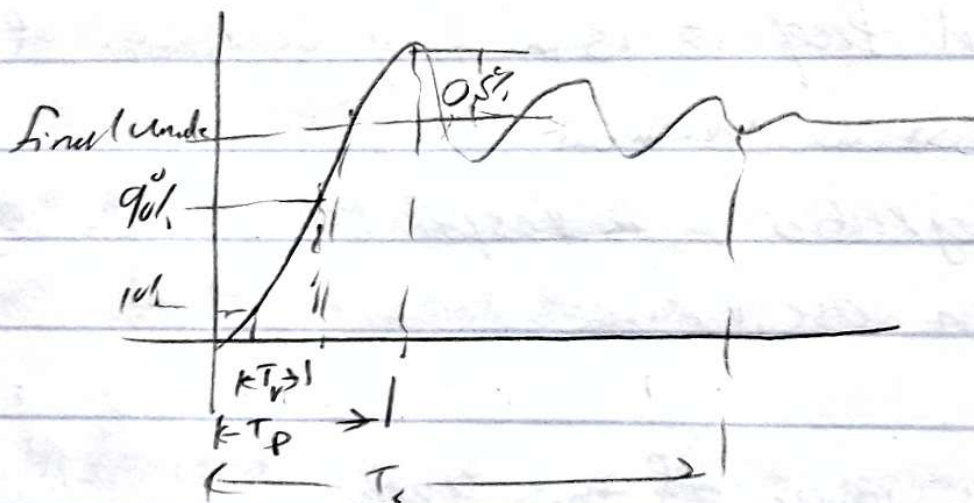
(1) $\xi = 0 \Rightarrow s_{1,2} = j\omega_n$ undamped

(2) $\xi = 1 \Rightarrow s_{1,2} = -\xi\omega_n$ critically.

(3) $\xi > 1 \Rightarrow s_{1,2} = -\xi\omega_n \pm \sqrt{\xi^2 - 1}$ over damped

(4) $0 < \xi < 1 \Rightarrow s_{1,2} = -\xi\omega_n \pm j\omega_n \sqrt{1 - \xi^2}$ under damped

* under damped Response for second order $0 < \xi < 1$



• (t_p) Peak time.

is the time required to reach to maximum value.

• (t_r) Rise time.

is the time required go from 10% to 90% from final

• (t_s) settling time.

is the time required for response to reach $\pm 2\%$ from final

• (0.5%) Percentage over shoot

0.5% = $\frac{\text{Maximum} - \text{Final}}{\text{Final}}$

$\times 100\%$

$$T_p = \frac{\pi}{\omega_n \sqrt{1-\zeta^2}}$$

$$OS\% = 100 e^{\frac{-\pi \zeta}{\sqrt{1-\zeta^2}}}$$

$$\zeta = \frac{-\ln\left(\frac{OS\%}{100}\right)}{\sqrt{\pi^2 + \ln^2\left(\frac{OS\%}{100}\right)}}$$

$$T_s = \frac{4}{\zeta \omega_n}$$

$$T_r = \frac{2.2}{\omega_n}$$

$$\omega_d = \omega_n \sqrt{1-\zeta^2} \quad \text{called damped freq.}$$

* Example: Find $\zeta, \omega_n, T_s, T_r, T_p, OS\%, K_{dc}$.

$$G(s) = \frac{361}{s^2 + 16s + 361}$$

$$K_{dc} = \lim_{s \rightarrow 0} G(s) = \frac{361}{361} = 1 \quad / \quad \omega_n = \sqrt{361} = 19$$

$$2\zeta\omega_n = 16 \Rightarrow \zeta = 0.421$$

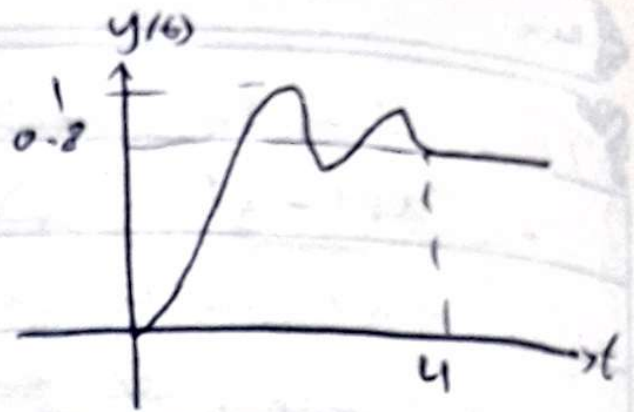
$$T_s = \frac{4}{\zeta \omega_n} = \frac{4}{0.421 \times 19} = 0.5 \text{ sec} \quad / \quad T_p = \frac{\pi}{\omega_n \sqrt{1-\zeta^2}} = 0.822 \text{ sec}$$

$$T_r = \frac{2.2}{\omega_n} = \frac{2.2}{19} = 0.115 \text{ sec}$$

$$OS\% = \frac{1 - \pi \zeta}{100 e^{\sqrt{1-\zeta^2}}} = 23.8\%$$

Example.

Find the transfer function from the step time response.



$$x(t) \text{ input} = \frac{t}{\tau} \Rightarrow y(t) \leq 1$$

$$\text{final value} = 0.8, \text{ input} = 1$$

$$S.S.E = 1 - 0.8 = 0.2.$$

$$K_{dc} = \frac{\text{final value}}{\text{input}} = \frac{0.8}{1} = 0.8.$$

$$T_s = 4 \text{ sec}, \text{ OS \%} = \frac{1 - 0.8}{0.8} = 25\%$$

$$\xi = \frac{-\ln\left(\frac{\text{OS \%}}{100}\right)}{\sqrt{\pi^2 + \ln^2\left(\frac{\text{OS \%}}{100}\right)}} = 0.403.$$

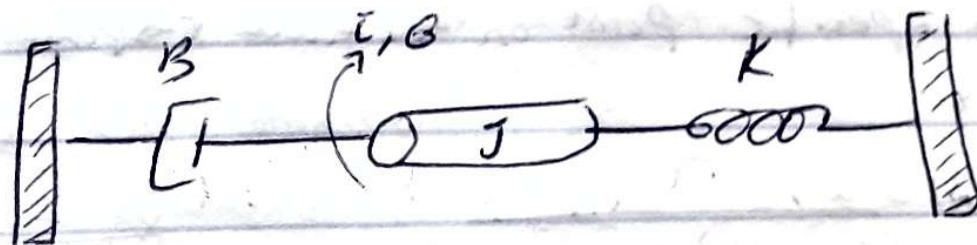
$$4 = \frac{4}{\delta \omega_n} \Rightarrow \omega_n = \frac{1}{0.403} = 2.481$$

$$G(s) = \frac{K_{dc} \omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

$$= \frac{4.92}{s^2 + 2s + 6.15}$$

$$K_{dc} = \frac{4.92}{6.15}$$

Example: Design this system to achieve this Point
 $\zeta = 0.5$, $T_s = 8 \text{ sec}$ & $K_{dc} = 1$.



$$T(s) = (Js^2 + Bs + K) \Theta(s).$$

$$\frac{\Theta(s)}{T(s)} = \frac{1}{Js^2 + Bs + K} \Rightarrow \frac{\Theta(s)}{T(s)} = \frac{V_J}{s^2 + \frac{B}{J}s + \frac{K}{J}}$$

$$T_s = 8 = \frac{4}{\zeta \omega_n} \Rightarrow \omega_n = \frac{4}{8 \times 0.5} = 1 \text{ rad/s.}$$

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \Rightarrow \frac{B}{J} = 2\zeta\omega_n$$

$$\frac{K}{J} = \omega_n^2.$$

$$\Rightarrow \frac{B}{J} = 2(0.5)(1) = 1.$$

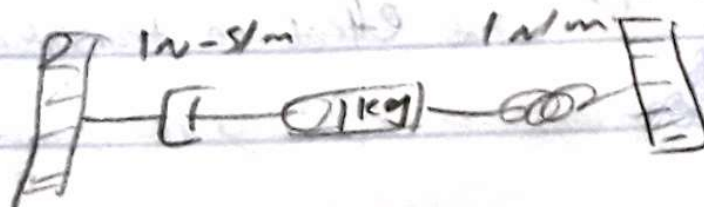
$$\frac{K}{J} = (1)^2 = 1$$

$$K_{dc} = 1 \Rightarrow K_{dc} \omega_n^2 = \frac{1}{J} \Rightarrow \omega_n^2 = \frac{1}{J}.$$

$$J = \frac{1}{\omega_n^2} = 1 \Rightarrow \frac{K}{J} = \omega_n^2 \Rightarrow K = J \omega_n^2 = 1.$$

$$\Rightarrow \frac{B}{J} = 1 \Rightarrow B = (1)(1) = 1.$$

$$G(s) = \frac{1}{s^2 + s + 1}$$

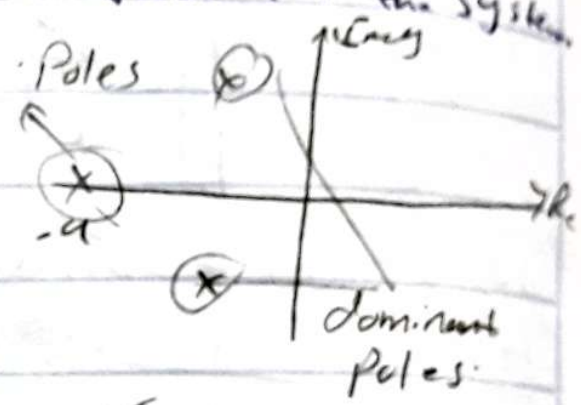


[4.7] System Response with additional Poles:

Now, we will be study the effect of adding poles at specific point on the time response for the system.

$$G(s) = \frac{a}{(s+a)(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

Additional Poles



* قاعده / اذا كنت تكتب Pole في مكان قريب من Dominant pole او Dominant pole يتحرك تأثيره على اننا نكتب تبعد عنها بعد ذلك اضعاف Real Part فهو يكون مهم ولا يوجد تأثير على شي فانه كانت قريبة ليعب بخاص مع هذا النوع من الان لا نكتب system يصح من ذلك اننا نكتب و بالتالي يكتب في دراسة قوانين اخرى.

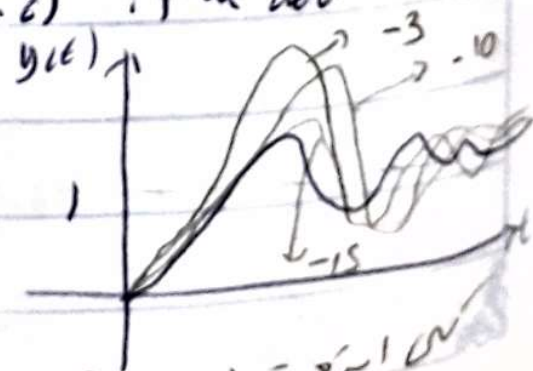
[4.8] System Response with Zeros:

→ Zero effect on OS%.

Ex: If you have 2 Poles at $(-1, \pm j2.2)$ if we add Zeros at

- 1) -3
- 2) -10
- 3) -15

what effect on transient response



dominant poles ...

... zeros

5 tubes
equation

output equation

* Eigen value $\Rightarrow$ to Find the poles of the system & it is equal the eigenvalues (2)

Ex: $\dot{X} = \begin{bmatrix} 0 & 2 \\ -3 & -5 \end{bmatrix} X + \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-t}$, $X(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$
 $y = [1 \ 3] X$

(a) solve state equation and $y(t)$.

(b) Find the eigen values and the system poles.

(a) $X(s) = (sI - A)^{-1} (X(0) + B U(s))$
 $= \begin{bmatrix} \frac{2s + \frac{2}{s} + 6}{s^2 + 5s + 6} \\ \frac{s - 5}{s^2 + 5s + 6} \end{bmatrix}$, $Y(s) = C X(s) + D U(s)$

$Y(s) = \frac{(5s + \frac{2}{s} - 3)}{s^2 + 5s + 6} = \frac{56/3}{s+3} - \frac{14}{s+2} + \frac{1/3}{s}$

$\Rightarrow y(t) = \left(\frac{56}{3} e^{-3t} - 14 e^{-2t} + \frac{1}{3} \right) u(t)$

(b) Eigen value $\det(sI - A) = 0$.

$s^2 + 5s + 6 = 0$

$s_1 = -2, s_2 = -3$

$\lambda_1 = -2, \lambda_2 = -3$

(4.11) Time domain solution.

• State transition matrix.

it is represent the free response (input = 0) of the system, defined the transition of the state from initial time at $t=0$ to any time (t) when the input equal zero.

$$\Phi(t) = \mathcal{L}^{-1} (sI - A)^{-1}$$

$$X(t) = \Phi(t) \cdot X(0) + \mathcal{L}^{-1} \{ \Phi(s) \cdot B U(s) \} \quad \text{Multiply.}$$

OR
$$X(t) = \Phi(t) X(0) + \int_0^t \Phi(t-\tau) B U(\tau) d\tau \quad \text{Convolution}$$

$$Y(t) = C X(t) + D U(t) \quad \#$$

* Examples:
$$X' = \begin{bmatrix} 0 & 2 \\ -2 & -5 \end{bmatrix} X + \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-2t}$$

$$Y = [0 \ 1] X, \quad X(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Find $X(t)$ and $Y(t)$.

$$\Phi(t) = \mathcal{L}^{-1} \{ (sI - A)^{-1} \}$$

$$(sI - A)^{-1} = \begin{bmatrix} s+5 & 2 \\ -2 & s \end{bmatrix} \cdot \frac{1}{s^2 + 5s + 4}$$

$$\Phi(t) = \begin{bmatrix} \frac{4}{3} e^{-4t} - \frac{1}{3} e^{-t} & \frac{2}{3} e^{-t} - \frac{2}{3} e^{-4t} \\ \frac{2}{3} e^{-4t} - \frac{2}{3} e^{-t} & \frac{4}{3} e^{-4t} - \frac{1}{3} e^{-t} \end{bmatrix}$$

Partial fraction

$$\Phi(t-\tau) = \begin{bmatrix} \frac{4}{3}e^{-(t-\tau)} & -\frac{1}{3}e^{-(t-\tau)} & \frac{2}{3}e^{-(t-\tau)} & -\frac{2}{3}e^{-4(t-\tau)} \\ -\frac{4}{3}e^{-(t-\tau)} & -\frac{1}{3}e^{-(t-\tau)} & \frac{4}{3}e^{-(t-\tau)} & -\frac{1}{3}e^{-(t-\tau)} \end{bmatrix}$$

$$B u(\tau) = \begin{bmatrix} 0 \\ e^{-2\tau} \end{bmatrix}_{2 \times 1}$$

$$\Phi(t-\tau) B u(\tau) = \begin{bmatrix} \left(\frac{2}{3}e^{-(t-\tau)} & -\frac{2}{3}e^{-4(t-\tau)} \right) e^{-2\tau} \\ \left(\frac{4}{3}e^{-(t-\tau)} & -\frac{1}{3}e^{-(t-\tau)} \right) e^{-2\tau} \end{bmatrix}$$

$$\int_0^t \Phi(t-\tau) B u(\tau) d\tau = \begin{bmatrix} \int_0^t \left(\frac{2}{3}e^{-(t-\tau)} & -\frac{2}{3}e^{-4(t-\tau)} \right) e^{-2\tau} d\tau \\ \int_0^t \left(\frac{4}{3}e^{-(t-\tau)} & -\frac{1}{3}e^{-(t-\tau)} \right) e^{-2\tau} d\tau \end{bmatrix}$$

تکرار

* Additional Exercise.

(i) 2nd order system settles to final position of 5 cm in a time of 2 seconds in response to unit step, the response of ~~2 seconds in resp~~ this system is oscillatory with damping freq of 3 rad/s, give the transfer function for this system.

Input = 1 cm, final value = 5 cm

$$K_{dc} = \frac{5}{1} = 5$$

$$T_s = 2 \text{ sec}, \omega_d = \omega_n \sqrt{1 - \xi^2} = 3 \text{ rad/s.}$$

$$\omega_d^2 = \omega_n^2 (1 - \xi^2) \Rightarrow \omega_n^2 - \xi^2 \omega_n^2$$

$$T_s = \frac{4}{\xi \omega_n} \Rightarrow T_s^2 = \frac{16}{\xi^2 \omega_n^2} \Rightarrow \xi^2 \omega_n^2 = \frac{16}{(2)^2} = 4$$

$$\Rightarrow \omega_d^2 = \omega_n^2 - \xi^2 \omega_n^2 \Rightarrow 4$$

$$(3)^2 = \omega_n^2 - 4 \Rightarrow \omega_n = \sqrt{13} = 3.6 \text{ rad/s.}$$

$$\xi = \frac{4}{2 \times 3.6} = 0.55$$

$$G(s) = \frac{K_{dc} \omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} = \frac{64.8}{s^2 + 3.96s + 13}$$

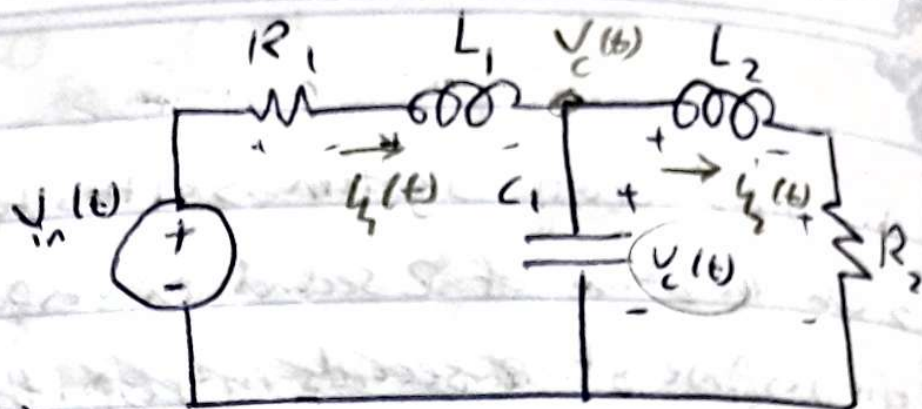
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(2) output $i_2(t)$.

(a) Find S.S.

(b) solve by using

time domain i_1, i_2, v_c .



$$V_{in}(t) = R_1 i_1(t) + L_1 \frac{di_1(t)}{dt} + v_c(t)$$

$$\Rightarrow \frac{di_1(t)}{dt} = -\frac{R_1}{L_1} i_1(t) - \frac{1}{L_1} v_c(t) + \frac{V_{in}(t)}{L_1}$$

$$v_c(t) = L_2 \frac{di_2(t)}{dt} + R_2 i_2(t)$$

$$\Rightarrow \frac{di_2(t)}{dt} = -\frac{R_2}{L_2} i_2(t) + \frac{v_c(t)}{L_2}$$

$$i_1(t) - i_2(t) = C \frac{dv_c(t)}{dt}$$

$$\Rightarrow \frac{dv_c(t)}{dt} = \frac{1}{C} i_1(t) - \frac{1}{C} i_2(t)$$

$$\begin{bmatrix} \dot{i}_1 \\ \dot{i}_2 \\ \dot{v}_c \end{bmatrix} = \begin{bmatrix} -\frac{R_1}{L_1} & 0 & -\frac{1}{L_1} \\ 0 & -\frac{R_2}{L_2} & \frac{1}{L_2} \\ \frac{1}{C} & -\frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ v_c \end{bmatrix} + \begin{bmatrix} \frac{1}{L_1} \\ 0 \\ 0 \end{bmatrix} V_{in}(t)$$

$$y = i_2(t) = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ v_c \end{bmatrix}$$

$$b) \quad x(t) = \phi(t) \cdot x(0) + \int_0^t \{ (S(t-\tau))^{-1} B u(\tau) \}$$

$\phi(t) \cdot x(0) = 0$ because no initial value.

Now, we need to find $\phi(t)$ or $\phi(s)$ &

make Convolution to solve this equation.

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = (sI - A)^{-1} B u(s)$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = (sI - A)^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\frac{1}{s^2 - 2s} \begin{bmatrix} 1 & 0 \\ 0 & s-2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} s-2 & 0 \\ 0 & s-2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

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$$\begin{bmatrix} s-2 & 0 \\ 0 & s-2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

3) The system is describe by :

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 1 \end{bmatrix}.$$

a) Evaluate the state transition matrix.

b) what is the response of the system due to input unit impulse and initial condition $x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

$$\phi(t) = \mathcal{L}^{-1} \{ (sI - A)^{-1} \}$$

$$(sI - A)^{-1} = \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix} \cdot \frac{1}{s^2 + 3s + 2}$$

Partial
fraction

$$\Rightarrow \phi(t) = \begin{bmatrix} -e^{-2t} + 2e^{-t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & 2e^{-2t} - e^{-t} \end{bmatrix}$$

$$x(s) = \underbrace{(sI - A)^{-1}}_n \left[\underbrace{x(0)}_1 + \underbrace{B u(s)}_2 \right], u(s) = 1$$

$$x(s) = \begin{bmatrix} s+4 \\ s-2 \end{bmatrix} \cdot \frac{1}{s^2 + 3s + 2}$$

$$y(s) = C x(s) + D u(s), D = 0.$$

$$\Rightarrow y(s) = \frac{2}{s+2}$$

$$\Rightarrow y(t) = 2e^{-2t} u(t)$$

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