

Chapter 3 Modeling in Time domain

TF $\rightarrow$ in chapter 2 for Linear system and time invariant system

State space Model:- in this chapter for Linear and non Linear system and for time invariant and time varying system and MIMO and SISO

In s.s. Model we convert n-th order DE's into n-first order differential equations

نحوته الأولى فقط -
من الدرجة الأولى فقط

State space equations

① State equations

$$\dot{X} = AX + Bu$$

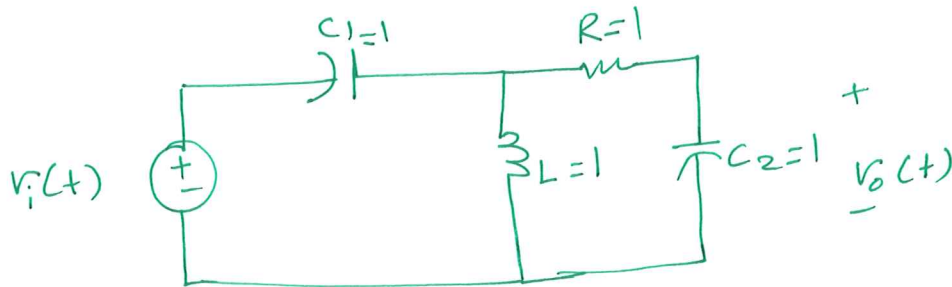
② output equations

$$y = CX + Du$$

the component of state vector must be linearly independent.

Assessment Exercise 3.1 :-

Find the s.s representation of electrical network. the output is $v_o(t)$



طريقة حل أى electrical s.s. :-

(1) عدد المعادلات المتفاضلة هو نفس عدد Storage element (C و L) طاقة .

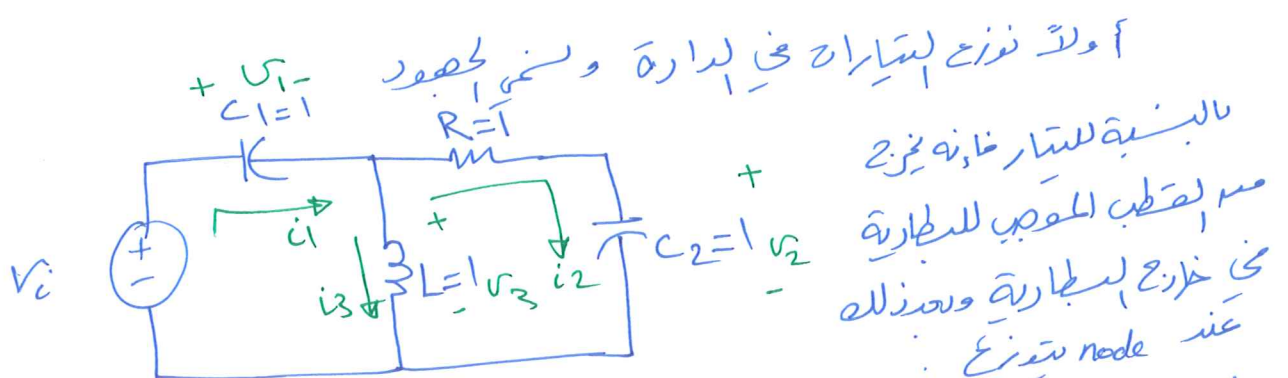
(2) أقاليمية للمعادلات المتفاضلة هي خارجة كل ملف ومكثف

$$v = L \frac{di}{dt} \quad \text{للملف}$$

$$i = C \frac{dv}{dt} \quad \text{للمكثف}$$

(3) بعد ذلك نحل المعادلات حيث أى معادلة تفاضلية تحتوي على المتغيرات أو على المدخل (input)

لنبدأ بالحل ...

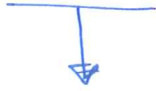


أولاً نوزع التيار في الدارة ونسحب الجهد بالنسبة للتيار خارجة من القطب الموجب للتيار في خارج التيارات وبعد ذلك عند node متفرع

أقاليمية للجهد ذاتها فيها ولقطبية تكون موجبة عند رأس التيارات أى أن التيار

يجب عند القطب المعكوب للـ λ
الآن نكتب المعادلات :-

$$v_3 = L \frac{di_3}{dt}, \quad i_1 = C_1 \frac{dv_1}{dt}, \quad i_2 = C_2 \frac{dv_2}{dt}$$

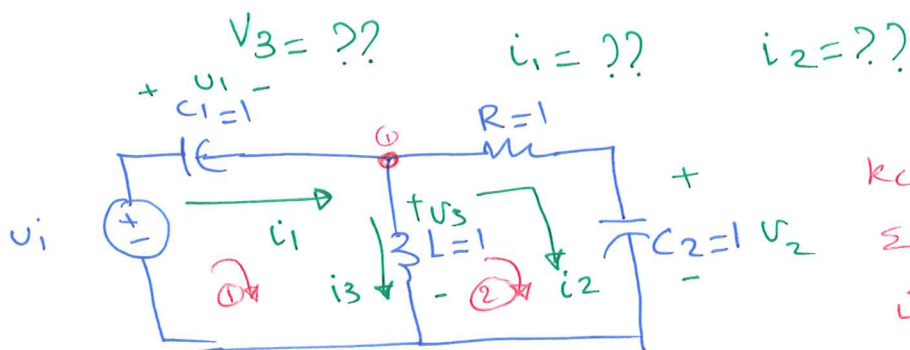


$$\frac{di_3}{dt} = \frac{v_3}{L}, \quad \frac{dv_1}{dt} = \frac{i_1}{C_1}, \quad \frac{dv_2}{dt} = \frac{i_2}{C_2}$$

الآن يمكننا كتابة كل المعادلات

$$\begin{bmatrix} \frac{dv_1}{dt} \\ \frac{dv_2}{dt} \\ \frac{di_3}{dt} \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ i_3 \end{bmatrix} + \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix} v_i$$

نلاحظ أنه بالمعادلات، المتغيرات تكون بالبنية i_3 و v_2 و v_1
أي صغرت آخر يكون غير مصرف لذلك نتجيب معادلات بدلا من هذه المقترحات.



KCL at node ①

$$\sum I_{in} = \sum I_{out}$$

$$i_1 = i_2 + i_3$$

KVL at ① ②

$$-v_i + v_1 + v_3 = 0$$

$$v_3 \text{ هنا } \boxed{v_3 = v_i - v_1}$$

KVL at ② ③

$$-v_3 + R i_2 + v_2 = 0$$

$$i_2 = \frac{V_3 - V_2}{R} = \frac{V_i - V_1 - V_2}{R}$$

$$i_1 = i_2 + i_3 = \frac{V_i - V_1 - V_2}{R} + i_3$$

$$\frac{di_3}{dt} = \frac{V_3}{L} = \frac{V_i - V_1}{L} = \frac{V_i}{L} - \frac{V_1}{L} \rightarrow (1)$$

$$\frac{dV_1}{dt} = \frac{i_1}{C_1} = \frac{V_i}{RC_1} - \frac{V_1}{RC_1} - \frac{V_2 + i_3 LC_1}{RC_1} \rightarrow (2)$$

$$\frac{dV_2}{dt} = \frac{i_2}{C_2} = \frac{V_i}{RC_2} - \frac{V_1}{RC_2} - \frac{V_2}{RC_2} \rightarrow (3)$$

$$\begin{bmatrix} \frac{di_3}{dt} \\ \frac{dV_1}{dt} \\ \frac{dV_2}{dt} \end{bmatrix} = \begin{bmatrix} -\frac{1}{RC_1} & -\frac{1}{RC_1} & \frac{1}{L} \\ -\frac{1}{RC_2} & -\frac{1}{RC_2} & 0 \\ -\frac{1}{L} & 0 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ i_3 \end{bmatrix} + \begin{bmatrix} \frac{1}{RC_1} \\ \frac{1}{RC_2} \\ \frac{1}{L} \end{bmatrix} V_i$$

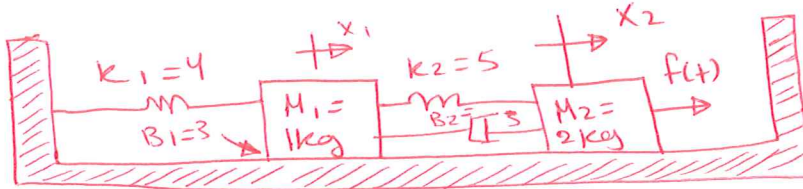
$$V_2 = V_0$$

substitute by $R = C_1 = C_2 = L = 1$

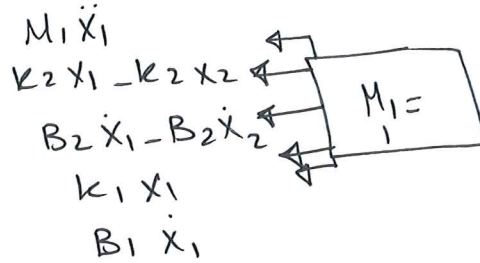
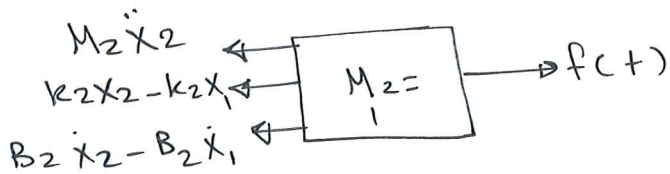
$$\begin{bmatrix} \frac{di_3}{dt} \\ \frac{dV_1}{dt} \\ \frac{dV_2}{dt} \end{bmatrix} = \begin{bmatrix} -1 & -1 & 1 \\ -1 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_0 \\ i_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} V_i$$

$$y = V_0(t) = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_0 \\ i_3 \end{bmatrix}$$

[2]



Free Body diagram



$$f(t) = M_2 \ddot{x}_2 + k_2 x_2 - k_2 x_1 + B_2 \dot{x}_2 - B_2 \dot{x}_1$$

$$M_1 \ddot{x}_1 + (B_1 + B_2) \dot{x}_1 + (k_1 + k_2) x_1 - k_2 x_2 - B_2 \dot{x}_2 = 0$$

بالبقوة مع القيم :-

$$f(t) = 2 \ddot{x}_2 + 5 x_2 - 5 x_1 + 3 \dot{x}_2 - 3 \dot{x}_1$$

$$\ddot{x}_1 + 6 \dot{x}_1 + 9 x_1 - 5 x_2 - 3 \dot{x}_2 = 0$$

نحسب كل معادلة متفرقة

$$U_1 = x_1$$

$$U_3 = x_2$$

$$V_2 = \dot{x}_1$$

$$V_4 = \dot{x}_2$$

$$f(t) = 2 \ddot{U}_4 + 5 U_3 - 5 U_1 + 3 \dot{U}_4 - 3 \dot{U}_2 \rightarrow 1)$$

$$\ddot{U}_2 + 6 \dot{U}_2 + 9 U_2 - 5 U_3 - 3 \dot{U}_4 = 0 \rightarrow 2)$$

$$\ddot{U}_2 = -6 \dot{U}_2 - 5 U_3 - 9 U_2 + 3 \dot{U}_4 \rightarrow 2^*)$$

$$\ddot{U}_4 = \frac{f(t)}{2} - \frac{5}{2} U_3 + \frac{5}{2} U_1 - \frac{3}{2} \dot{U}_4 + \frac{3}{2} \dot{U}_2 \rightarrow 1^*)$$

$$\begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \\ \dot{v}_3 \\ \dot{v}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -9 & -6 & 5 & 3 \\ 0 & 0 & 0 & 1 \\ \frac{5}{2} & \frac{3}{2} & -\frac{5}{2} & -\frac{3}{2} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{bmatrix} f$$

Converting transfer function to state space

① Direct decomposition

is applied to TF that is not factored form

② Cascade (series) decomposition

is applied to TF that is written as a product of simple first or 2nd order components (factor)

③ Parallel decomposition

have a common input and an output formed

By the algebraic sum of the outputs from all the subsystems.

3.11 use direct Method. ~~from~~ TF to s.s
 ↓
 convert

نفر. تابع $\frac{C(s)}{R(s)}$ لعدد $\frac{1}{s^4}$

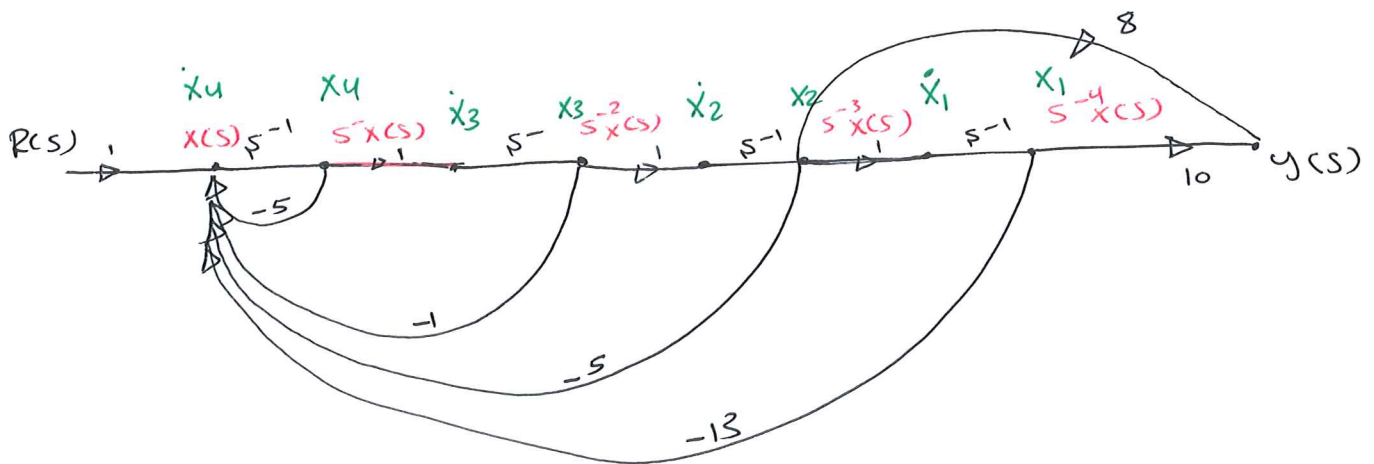
$$\frac{C(s)}{R(s)} = \frac{8s + 10}{s^4 + 5s^3 + s^2 + 5s + 13} \cdot \frac{s^{-4}}{s^{-4}} \frac{X(s)}{X(s)}$$

$$C(s) = (8s + 10) s^{-4} X(s) \\ = 8s^{-3} X(s) + 10s^{-4} X(s)$$

درجه المقام هي نفس عدد
 المتكامل (integrator) وهو
 نفس عدد المتغيرات.

$$R(s) = (s^4 + 5s^3 + s^2 + 5s + 13) s^{-4} X(s) \\ = X(s) + 5s^{-1} X(s) + s^{-2} X(s) + 5s^{-3} X(s) + 13s^{-4} X(s)$$

$$X(s) = R(s) - 5s^{-1} X(s) - s^{-2} X(s) - 5s^{-3} X(s) - 13s^{-4} X(s)$$



$$\dot{X}_1 = X_2$$

$$\dot{X}_2 = X_3$$

$$\dot{X}_3 = X_4$$

$$\dot{X}_4 = R - 5X_4 - X_3 - 5X_2 - 13X_1$$

$$Y = 8X_2 + 10X_1$$

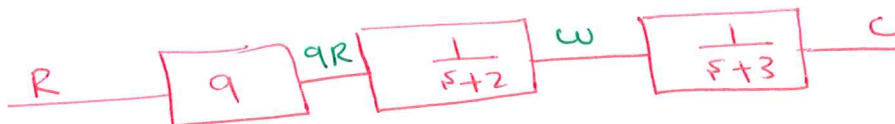
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -8 & -5 & -1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} R$$

$$y = \begin{bmatrix} 10 & 8 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

* use cascade decomposition

$$G(s) = \frac{9}{s^2 + 5s + 6} = \frac{9}{(s+2)(s+3)} = \frac{C(s)}{R(s)}$$

2 integrators $\leftarrow$



$$\frac{w}{9R} = \frac{1}{s+2} \cdot \frac{s^{-1} u_1}{\frac{s^{-1} u_1}{s^{-1} u_1}}$$

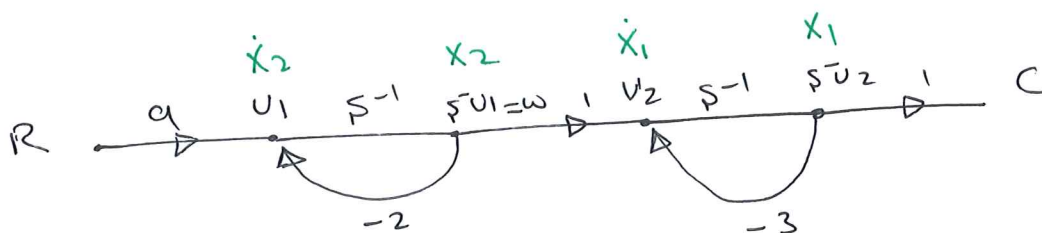
$$w = s^{-1} u_1$$

$$9R = u_1 + 2s^{-1} u_1 \rightarrow u_1 = 9R - 2s^{-1} u_1$$

$$\frac{C}{w} = \frac{1}{s+3} \cdot \frac{s^{-1} u_2}{\frac{s^{-1} u_2}{s^{-1} u_2}}$$

$$C = s^{-1} u_2$$

$$w = u_2 + 3s^{-1} u_2 \rightarrow u_2 = w - 3s^{-1} u_2$$



$$\dot{x}_1 = -3x_1 + x_2$$

$$\dot{x}_2 = 9R - 2x_2$$

$$C = x_1$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 9 \end{bmatrix} R$$

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

* Use parallel decomposition (partial fraction)

Very important Note:- poles must be real.

$$\text{ex) } G(s) = \frac{s+1}{s^3+9s^2+26s+24} = \frac{s+1}{(s+2)(s+3)(s+4)}$$

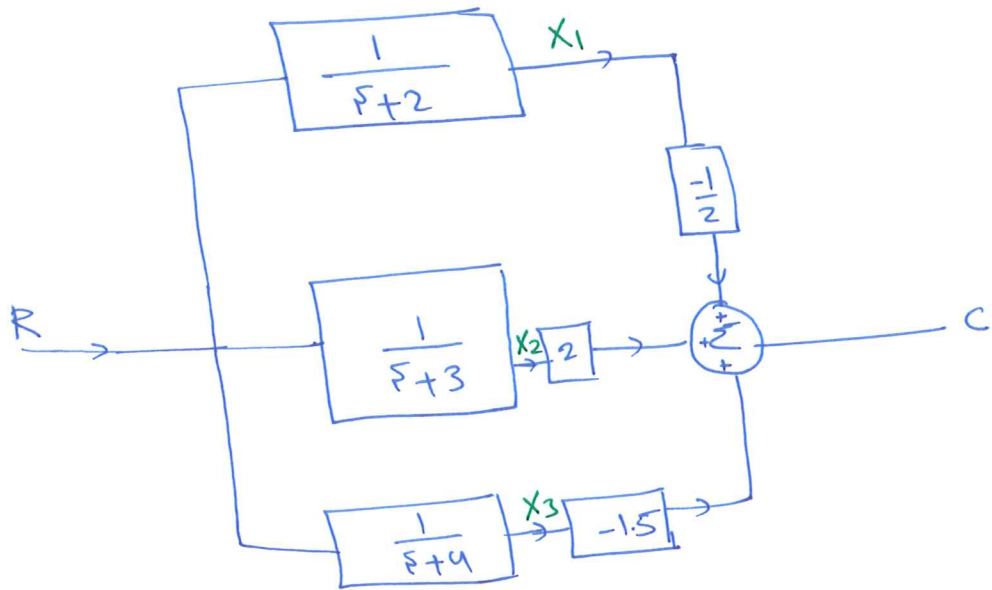
$$\frac{(s+1)}{s^3+9s^2+26s+24} = \frac{A}{s+2} + \frac{B}{s+3} + \frac{C}{s+4}$$

$$A = \frac{(s+2)(s+1)}{(s+2)(s+3)(s+4)} \Big|_{s=-2} = \frac{-1}{1 \times 2} = -\frac{1}{2}$$

$$B = \frac{(s+3)(s+1)}{(s+2)(s+3)(s+4)} \Big|_{s=-3} = \frac{-2}{-1 \times 1} = 2$$

$$C = \frac{(s+4)(s+1)}{(s+2)(s+3)(s+4)} \Big|_{s=-4} = \frac{-3}{2 \times 1} = -3/2$$

$$G(s) = \frac{C(s)}{R(s)} = \frac{-\frac{1}{2}}{s+2} + \frac{2}{s+3} + \frac{-3/2}{s+4}$$



$$\frac{x_1}{R} = \frac{1}{s+2} \cdot \frac{s^{-1} u_1}{\frac{s^{-1}}{u_1}} \Rightarrow$$

$$x_1 = s^{-1} u_1$$

$$u_1 = R - 2s^{-1} u_1$$

$$\frac{x_2}{R} = \frac{1}{s+3} \cdot \frac{s^{-1} u_2}{\frac{s^{-1}}{u_2}} \Rightarrow$$

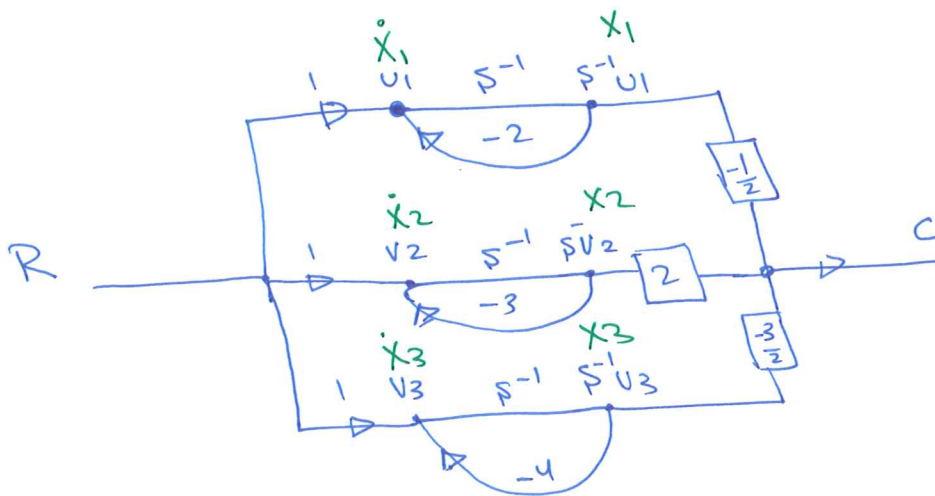
$$x_2 = s^{-1} u_2$$

$$u_2 = R - 3s^{-1} u_2$$

$$\frac{x_3}{R} = \frac{1}{s+4} \cdot \frac{s^{-1} u_3}{\frac{s^{-1}}{u_3}} \Rightarrow$$

$$x_3 = s^{-1} u_3$$

$$u_3 = R - 4s^{-1} u_3$$



$$\dot{x}_1 = R - 2x_1$$

$$\dot{x}_2 = R - 3x_2$$

$$\dot{x}_3 = R - 4x_3$$

$$C = -\frac{1}{2}x_1 + 2x_2 - \frac{3}{2}x_3$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} R$$

$$y = \left[-\frac{1}{2} \quad 2 \quad -\frac{3}{2} \right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Converting from s.s to TF.

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \rightarrow$$

ss in T domain

$$C(sI - A)^{-1}B + D$$

TF in f domain

Q:- Convert from s.s. to TF

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -4 & -1.5 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u(t)$$

$$y = \begin{bmatrix} 1.5 & 0.625 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$G(s) = C(sI - A)^{-1}B + D$$

$$sI - A = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -4 & -1.5 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} s+4 & 1.5 \\ -4 & s \end{bmatrix}$$

$$(sI - A)^{-1} = \frac{\text{adj}(sI - A)}{\det(sI - A)}$$

$$\text{adj}(sI - A) = \begin{bmatrix} s & -1.5 \\ 4 & s+4 \end{bmatrix}$$

$$\det(sI - A) = s(s+4) + 6 = s^2 + 4s + 6$$

$$(sI - A)^{-1} = \frac{\begin{bmatrix} s & -1.5 \\ 4 & s+4 \end{bmatrix}}{s^2 + 4s + 6}$$

$$C(sI - A)^{-1} = [1.5 \quad 0.625] \begin{bmatrix} s & -1.5 \\ 4 & s+4 \end{bmatrix} \cdot \frac{1}{s^2 + 4s + 6}$$

$$[1.5s + 2.5 \quad 0.625s + 0.25] \cdot \frac{1}{s^2 + 4s + 6}$$

$$C(sI - A)^{-1}B = [1.5s + 2.5 \quad 0.625s + 0.25] \cdot \begin{bmatrix} 2 \\ 0 \end{bmatrix} \cdot \frac{1}{s^2 + 4s + 6}$$

$$= \frac{3s + 5}{s^2 + 4s + 6}$$

Linearization

Final 2005:-)

A plant is described by the following differential equations:-

$$\dot{x}_1 = 3x_1 + \sin(x_2) + 5u$$

$$\dot{x}_2 = 4\sin x_1 + 2u \cos x_1 + 3x_2^2$$

Find the linearized state model approximation for the above system

about system nominal point $x_{01} = x_{02} = u_0 = 0$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 3 & \cos x_2 \\ 4\cos x_1 - 2u \sin x_1 & 6x_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 5 \\ 2\cos x_1 \end{bmatrix} u$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 3 & \cos 0 = 1 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 5 \\ 2 \end{bmatrix} u$$

Midterm 2003:- Find the linearized approximation about nominal point $x_0 = 0$

$$\dot{x}_1 - x_2^2 - x_1 \cos x_2 = 0$$

$$\dot{x}_2 - x_2 - (x_1 + 1)x_1 - x_1 \sin x_2 = 0$$

Solution:-

$$\dot{x}_1 = x_2^2 + x_1 \cos x_2$$

$$\dot{x}_2 = x_2 + (x_1 + 1)x_1 + x_1 \sin x_2$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \cos x_2 & 2x_2 - x_1 \sin x_2 \\ 2x_1 + 1 + \sin x_2 & 1 + x_1 \cos x_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$