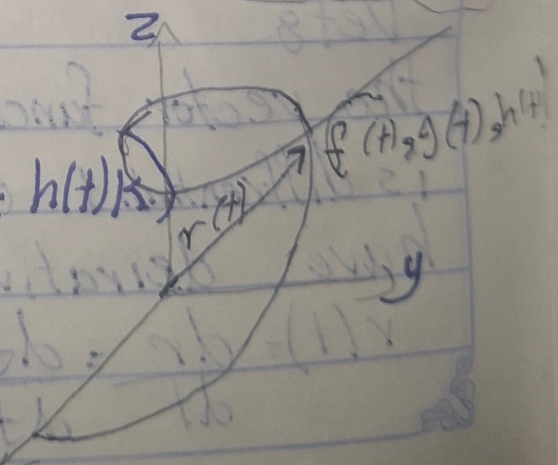
Ch. 13 × vector-valued function and motion n space

13.1 Curves in Space and their tangents
(المنحنيات في الفراغ والمماسات في الفراغ)

Def:- ① A curve in space is the partical's path that difiend by the points
$(x, y, z) = (f(t), g(t), h(t))$
I (I is a time interval)

② the poisition vector of a partical moves in space is $r(t) = \vec{OP} = f(t)i + g(t)j + h(t)k$
$t \leftarrow I$
this vector is called the vector values function

Limits and comunity:
Def: if $r(t) = f(t)i + g(t)j + h(t)k$
is a vector valid function then
$\lim_{t \to t_0} r(t)$ is defind by



التاريخ: / / اليوم: الدرس:

$$\lim_{t\to t_0} r(t) = \lim_{t\to t_0}\left(f(t)\,i + g(t)\,j + h(t)\,K\right)$$

$$= \lim_{t\to t_0} f(t)\,i + \lim_{t\to t_0} g(t)\,j + \lim_{t\to t_0} h(t)\,K$$

$$= L_1 i + L_2 j + L_3 K = \vec{L}$$

$\vec{L}$ constant vector

L Limit ∥ exist ∦ not exist

Def

A vector function $r(t)$ is continous at a point $t = t_0$ in its domain if $\lim_{t\to t_0} r(t) = r(t_0)$

also $r(t) = f(t)\,i + g(t)\,j + h(t)\,K$ is continous at $t = t_0$ if f, g and h are continous at $t = t_0$

Def

the vector function $r(t) = f(t)\,i + g(t)\,j + h(t)\,K$ is diffrenthabi. at t if f, g and h have deiratives at t and

$$r'(t) = \frac{dr}{dt} = \frac{df}{dt}\,i + \frac{dg}{dt}\,j + \frac{dh}{dt}\,K$$

vector ← list (متجه)

متجه (تحليل جبري)

Def:

The curve $r(t)$ is called smooth (منحنى أملس) if $\frac{dr}{dt}$ is continous (متصل) and $\frac{dr}{dt} \neq 0$ that is if f, g and h have continous (متصل) first derivative (مشتقات أولى) and $\frac{df}{dt}, \frac{dg}{dt}, \frac{dh}{dt}$ not all zero (ما يساوي صفر)

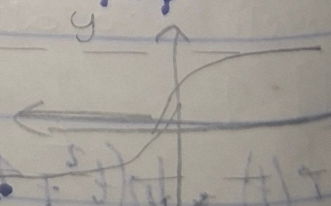
* تكون الدالة منصلة إذا كانت قيمة الدالة عند تلك النقطة تساوي نهاية الدالة عند تلك النقطة

Notes:

On a smooth curve there is neighter cordor and cusp

Example: Let $r(t) = e^{-t}i + \left(\frac{t-1}{t+1}\right)j + \tan^{-1}t\,k$

Find $\lim_{t\to\infty} r(t)$



Solution

$$\lim_{t\to\infty} r(t) = \lim_{t\to\infty}\frac{1}{e^t}i + \lim_{t\to\infty}\left(\frac{t-1}{t+1}\right)j + \lim_{t\to\infty}\tan^{-1}k$$

$$0i + j + \frac{\pi}{2}k$$

Example ② the vector function $r(t) = \cos(t)\,i + \sin(t)\,j + t\,K$ is continuous?

Solution

its continuous $\forall t$ because of cost, sin t and t is continuous $\forall t$

* the vector function $r(t) = \cos t\,i + \sin t\,j + [\![t]\!]K$ floor function is continuous at every integrar because $\lim_{t \to c} [\![t]\!]$ does not exist, c is an integar

$\Rightarrow$ If $r(t) = \ln(t^2+1)\,i + \tan^{-1} j + te^t K$

find $\vec{r}'(1)$

Solution $\Rightarrow \frac{dr}{dt} = \frac{1}{t^2+1} \times 2t\,i + \frac{1}{1+t^2} j +$

$(e^t + te^t)K$

*Example: Is the curve $r(t) = t^3 i + (3t^2 - 2t) j + t^2 K$ smooth?

continuous ✓

$\neq 0$ ✓

Solution:

$\vec{r}'(t) = 3t^2 i + (6t - 2) j + 2t K$

its function $3t^2$, $6t - 2$ and $2t$ are continuous for t because they are polynomial and not all zero at any point.

So the given curve $r(t) = t^3 i + (3t^2 - 2t) j + t^2 K$ is smooth.

Def:

If $r(t)$ is the position vector of a particle moving a smooth curve in space then

① $V(t) = \frac{dr}{dx}$ is called the particle's velocity vector that is tangent to the curve

② $|V|$ is called the paticle's speed

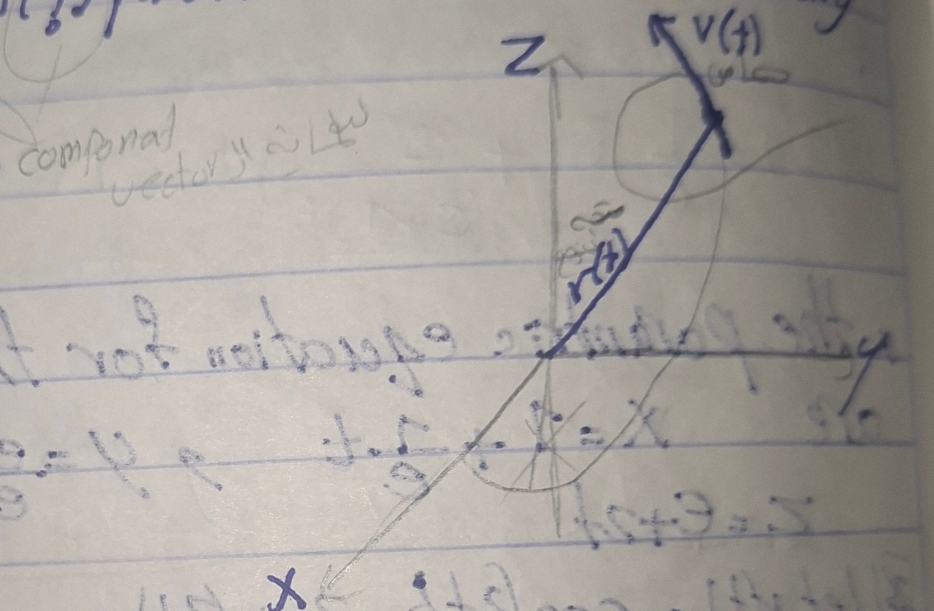
③ $\frac{V}{|V|}$ is the direction of the velocity and the direction of the maition

④ $a = \frac{dv}{dt} = \frac{d^2r}{dt^2}$ is the particles acceleration vector (تعجيل)

Remark:- for a smooth curve the velocity is never zero ⇒ The partical does not stop and does not reverse its direction

خط المماس

Def: The tangent line to a smooth curve $r(t) = f(t)i + g(t)j + h(t)K$ at $t = t_0$ is the line that passes thraigh the point $(f(t_0), g(t_0), h(t_0))$ parallel $v(t_0)$, the velocity vector at t_0



Example: ~~The point $(f(e), g(e), h(e))$~~
$= (\ln e, e-1$

Find the paramitric equation for the line that is tangent to the curve

$r(t) = \ln t\, i + \left(\frac{t-1}{t+1}\right)j + t \ln t\, K$ at $t=e$

The point $(f(e), g(e), h(e))$

$= (\ln e, \frac{e-1}{e+1}, e \ln e)$

$(1, \frac{e-1}{e+1}, e)$ lie on the line

$$V(t) = \frac{dr}{dt} = \frac{1}{t} i + \frac{t+1-t+1}{(t+1)^2} + (1+\ln t) K$$

$$V(e) = \frac{1}{e} i + \frac{2}{(e+1)^2} j + 2K$$

* the parametric equation for the tangent line are $x = 1 + \frac{1}{e}t$, $y = \frac{e-1}{e+1} + \frac{2}{(e+1)^2} t$

$z = e + 2t$

(a) let $r(t) = \cos(2t) i + 3\sin(2t) j$

find the partical path interms of x and y

(b) find the particales velocity and acceleration vectors at $t = \frac{\pi}{6}$

Solutions

$x = \cos 2t$, $y = 3\sin 2t$

$x = \cos 2t$, $\frac{y}{3} = \sin 2t$

$$x^2 + \left(\frac{y}{3}\right)^2 = \cos^2 2t + \sin^2 2t$$

a) $\frac{x^2}{1} + \frac{y^2}{4} = 1$

ellips with center at the origin and the major axis along the y-axis.

b) $v(t) = \frac{dr}{dt} = -2\sin 2t\, i + 6\cos 2t\, j$

$v\left(\frac{\pi}{6}\right) = -2\sin\frac{\pi}{3}\, i + 6\cos\frac{\pi}{3}\, j$

$= -\sqrt{3}\, i + 3j$

$a = \frac{dv}{dt} = -4\cos 2t\, i - 12\sin 2t\, j$

$a\left(\frac{\pi}{6}\right) = -4\cos\frac{\pi}{3}\, i - 12\sin\frac{\pi}{3}\, j$

$= -2\, i - 6\sqrt{3}\, j$

Examples let $v(t) = \ln(t^2+1)\,i + \tan^{-1}t\,j + \sqrt{t^2+1}\,k$ At $t=1$ find

(a) the particeles velocity and acceleration

(b) the speed and the direction of motion

(c) the angle between velocity and acceleration vector

solution:

(a) $v(t) = \frac{dv}{dt} = \frac{2t}{t^2+1}\,i + \frac{1}{1+t^2}\,j + \frac{t}{\sqrt{t^2+1}}\,k$

$v(1) = \frac{2}{1^2+1}\,i + \frac{1}{1+1^2} + \frac{1}{\sqrt{1+1}}\,k$

$+\frac{1}{2\sqrt{t^2+1}} \cdot 2t$

$= 1 + \frac{1}{2}\,j + \frac{1}{\sqrt{2}}\,k$

$a = \frac{dv}{dt} = \frac{2(t^2+1) - 2t(2t)}{(t^2+1)^2}\,i - \frac{2t}{(1+t^2)}\,j + \frac{\sqrt{t^2+1} - t}{t^2+1}\,k$

$\frac{-2t^2+2}{(t^2+1)^2}\,i - \frac{2t}{(1+t^2)}\,j + \frac{1}{(t^2+1)\sqrt{t^2+1}}\,k$

$a(1) = -\frac{1}{2}\,j + \frac{1}{2\sqrt{2}}\,k$

التاريخ: / / اليوم:

(b) speed $|v| = \sqrt{1 + \frac{1}{4} + \frac{1}{2}} = \sqrt{\frac{7}{4}} = \frac{\sqrt{7}}{2}$

direction motion $= \frac{v}{|v|} = \frac{i + \frac{1}{2}j + \frac{1}{2}k}{\frac{\sqrt{7}}{2}}$

$= \frac{2}{\sqrt{7}}i + \frac{1}{\sqrt{7}}j + \frac{2}{\sqrt{14}}k$

(c) $a.v = |a||v|\cos\phi$

$0 - \frac{1}{4} + \frac{1}{4} = \sqrt{\frac{1}{4} + \frac{1}{8} + \frac{1}{4} + \frac{1}{2}} = \sqrt{1 + \frac{1}{4} + \frac{1}{4}}\cos\phi$

$\cos\phi \Rightarrow 0 \Rightarrow \phi = \frac{\pi}{2}$ the velocity and acceleration are orthogonal at $t = 1$

* Difrentiation Rules قواعد الاشتقاق

let u, v be diff vector fn. of t

C a constant vector, c any scalar and f any diff scalar fn then

(1) $\frac{d}{dt}(C) = 0$

(2) $\frac{d}{dt}(cu) = c\dot{u}(t)$

and $\frac{d}{dt}(fu) = f(t)u + f(t)\frac{du}{dt}$

③ $\frac{d}{dt}(u \pm v) = \dot{u}(t) \pm \dot{v}(t)$

④ $\frac{d}{dt}(u.v) = \dot{u}.v + u.\dot{v}$ التبديل ليس ضروري

⑤ $\frac{d}{dt}(u \times v) = \frac{du}{dt} \times v * v + u \times \frac{dv}{dt}$ التبديل ضروري

⑥ $\frac{d}{dt}(u\,f(t)) = \dot{u}\,f(t)\,f'(t)$ chain Rule قاعدة السلسلة

*vector fun of a constant length

Remarks:

If $r(t)$ is a diff vector fun of t and $r(t)$ is of constant length then $r(t).\frac{dr}{dt} = 0$ (means that $r(t)$ and $\frac{dr}{dt}$ are orthogonal)

Proof

$r(t).r(t) = |\vec{r}(t)| = C$

$r(t).\frac{dr}{dt} + \frac{dr}{dt}.r(t) = 0$

$2r(t).\frac{dr}{dt} = 0 \Rightarrow \boxed{r(t).\frac{dr}{dt} = 0}$

Examples

show that the vector $r(t) = 5\cos t\,i + 5\sin t\,j + 12k$ has a constant length and is orthogonal to its derivative

solution :-

$$|r(t)| = \sqrt{25\cos^2 t + 25\sin^2 t + 144}$$

$$= \sqrt{25(\cos^2 t + \sin^2 t) + 144} = \sqrt{169} = 13$$

So the given vector is of constant

$$\frac{dr}{dt} = -5\sin t\,i + 5\cos t\,j + 0$$

$$r(t) \cdot \frac{dr}{dt} = -25\sin t\cos t + 25\sin t\cos t$$

So $r(t)$ and $\frac{dr}{dt}$ are orthogonal

(13.2) Integrals of vector function

Defn.

① Indefinite integral $\Rightarrow$ If $r(t) = f(t)\,i + g(t)\,j + h(t)\,K$ where f, g and h are integrable fn.s of them

$$\int r(t)\,dt = \int (f(t)\,dt)\,i + \int (g(t)\,dt)\,j + \int (h(t)\,dt)\,K + C$$

where C is a constan vector

② definite vector إذا كانت الدالة متصلة على فترة مغلقة

$$\int_a^b r(t)\,dt = \int_a^b (f(t)\,dt)\,i + \int_a^b (g(t)\,dt)\,j + \int_a^b (f(t)\,dt)\,K$$

Examples سؤال 10 ص الكتاب

Evaluate the following intergral

$$\int_0^{\pi} \sec t\, i + \tan^2 t\, j - t \sin t\, K$$

$$= \int_0^{\pi} [\sec t\, i + (\sec^2 t - 1) j - t \sin t]\, k$$

$$= \ln|\sec t + \tan t|_0^{\pi} i + (\tan t - t)_0^{\pi} j - (\tan t$$

$$= (\ln|-1-0| - \ln|1+0|) i + (0 - \pi - 0 + 0) j + (\pi(-1) + 0 - 0 - 0) k$$

Derivative | Integral

t → $+$ → $\sin t$

1 → $-$ → $-\cos t$

0 → $-\sin t$

$$= 0 - \pi j - \pi k = -\pi(j + k)$$

Example (مثال ص 16 من الكتاب)

if $\vec{r}(t) = -(i + j + k)$, $r(0) = 10i + 10j + 10k$ and $\frac{dr}{dt}$ at $t = 0$ = zero

find $r(t)$??

Solutions

$$\int r''(t)\,\frac{dr}{dt} = \int(-i - j - K)\,dt$$

$$\int \frac{dr}{dt} = r'(t) = -ti - tK - tK + C_1$$

$$r'(0) = 0 \longrightarrow 0i - 0i - 0K + C_1 \Rightarrow \boxed{C_1 = 0}$$

$$\frac{dr}{dt} = -ti - tj - tK$$

$$r(t) = -\frac{1}{2}t^2 i - \frac{1}{2}t^2 j - \frac{1}{2}t^2 K + C_2$$

$$r(0) = 0 + C_2 = 10i + 10j + 10K$$

$$C_2 = 10i + 10j + 10K$$

$$r(t) = -\frac{1}{2}t^2 i - \frac{1}{2}t^2 j - \frac{1}{2}t^2 K + 10i + 10j + 10K$$

$$\boxed{= \left(10 - \frac{1}{2}t^2\right)i + \left(10 - \frac{1}{2}t^2\right)j + \left(10 - \frac{1}{2}10^2\right)K}$$

(b-16)

حل صعوبة الواجب

طول القوس في الفراغ

13.3 Arc Length in Space

Arc length along a space Curve

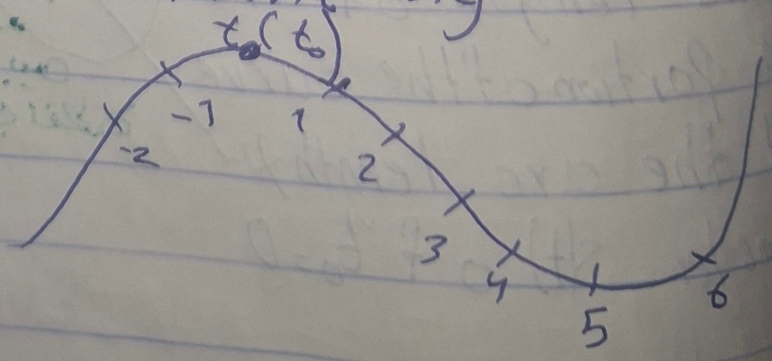
Def:-
the length of a smooth curve $r(t) = x(t)i + y(t)j + z(t)k$, $a \le t \le b$ as t increases from $t=a$ to $t=b$ is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

$$= \int_a^b |v|\, dt$$

تكامل من a إلى b

Note:-
the smooth curve is scalad like number like as shown in the fig

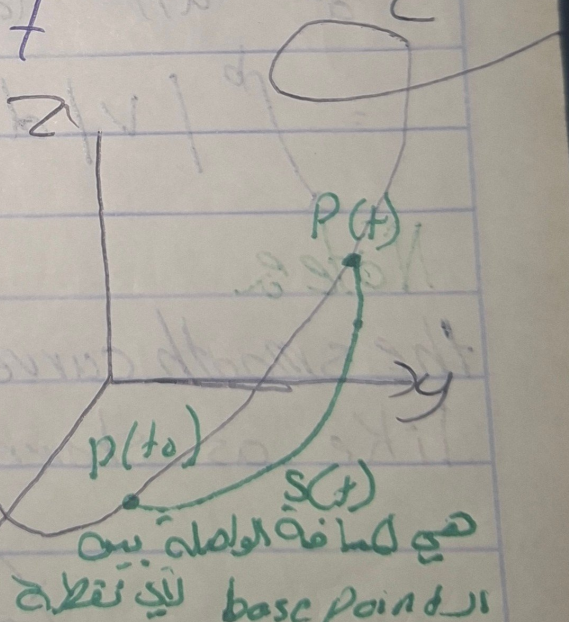


Defs.

the directed distance along a smooth curve C from the base point is

$$s(t) = \int_{t_0}^{t} |v(\tau)|\,d\tau = \int_{t_0}^{t} \sqrt{(x'(\tau))^2 + (y'(\tau))^2 + z'(\tau)^2}\,d\tau$$

Also, $s(t)$ is called the arclength parameter of the curve C which is increases in the direction of increasing t



Example:-

let $r(t) = e^t \cos t\, i + e^t \sin t\, j + e^t k$, $-\ln 4 \le t \le 0$

(a) find the length of the given portion of the curve

(b) find the arc length parameter $s(t)$ of $t_0 = 0$

التاريخ: / / اليوم:

المعادلة البارامترية

solution :- لحساب طول المنحنى يجب أولاً معرفة المشتقة الأول بالنسبة لـ t

(a) $\frac{dx}{dt} = \frac{d}{dt}(e^t \cos t)$ حاصل ضرب دالتين

$(e^t \cos t - e^t \sin t) = e^t(\cos t - \sin t)$

$\left(\frac{dx}{dt}\right)^2 = e^{2t}(\cos^2 t - 2\sin t \cos t + \sin^2 t)$

$= e^{2t}(1 - \sin t \cos t)$

$\frac{dy}{dt} = e^t \sin t + e^t \cos t = e^t(\sin t + \cos t)$

$\left(\frac{dy}{dt}\right)^2 = e^{2t}(\sin^2 t + 2\sin t \cos t + \cos^2 t) = e^{2t}(1+$

ثم $L = \int_{-\ln 4}^{0} \sqrt{e^{2t}(1 - 2\sin t \cos t)(1 + 2\sin$ $+ e^{2t}}\, dt$

$= \int_{-\ln 4}^{0} \sqrt{e^{2t}(2)} + e^{2t}\, dt$

$= \sqrt{3} \int_{-\ln 4}^{0} e^t\, dt = \sqrt{3} e^t \Big]_{-\ln 4}^{0}$ (a)

$= \sqrt{3}\left[1 - e^{-\ln 4}\right] = \sqrt{3}\left[1 - \frac{1}{4}\right] =$

(b) $s(t) = \int_0^t |v(\tau)|\, d\tau = \int_0^t \sqrt{3}\, e^{\tau}\, d\tau$,

$= \sqrt{3}\, e^{\tau}\Big|_0^t = \sqrt{3}(e^t - e^0) = \boxed{\sqrt{3}(e^t - 1)}$

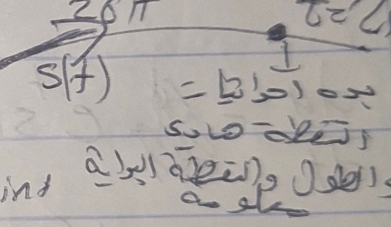
Example: find the point on the curve $r(t) = 5\sin t\, i + 5\cos t\, j + 12t\, k$ at a distance 26π units along the curve from the point $(0, 5, 0)$ in the direction of increasing arc length.

Solutions:

$x = 5\sin t = 0 \Rightarrow t = 0, \pi, 2\pi$ ($\sin = 0$)

$y = 5\cos t = 5 \Rightarrow t = 0, 2\pi$ ($\cos = 1$)

$z = 12t = 0 \rightarrow t = 0$



كيف بجيبها؟ $s(t)$

بروح بتفرق الدالة اللي كنت

وبجيب المركبة = تفرق المركبة

$v(t) = \frac{dr}{dt} = 5\cos t\, i - 5\sin t\, j + 12k$

$|v| = \sqrt{25\cos^2 t + 25\sin^2 t + 144} = \sqrt{25 + 144}$

$= \sqrt{169} = 13$

$s(t) = \int_0^t |v|\,|\tau|\, d\tau = \int_0^t 13\, d\tau = 13\tau\Big|_0^t = \boxed{13t}$

$t = 2\pi \Rightarrow t = 2\pi$

the point $(x, y, z) = (5\sin 2\pi, 5\cos 2\pi, 12(2\pi))$

$= (0, 5, 24\pi)$

Example find the length of the curve $r(t) = t^2 i + 2t j + \ln t\, k$ from the point $(1, 2, 0)$ to $(e^2, 2e, 1)$ where $t > 0$

لازم تكون قيمة t أكبر من zero
لانها قيمة الداخل المقام

solution :-

$(x, y, z) = (e^2, 2t, \ln t)$

at the point $(1, 2, 0)$

~~at the point~~ we have

$t = 1$ $t = 1$

$t^2 = 1$, $2t = 2$

$\ln t = 0$ $t = 1$

$t = 1$

$t^2 = e^2 \Rightarrow t = e$, $2t = 2e \Rightarrow t = e$

$\ln t = 1 \Rightarrow t = e$ $\boxed{t = e}$

$v(t) = 2t\,i + 2j + \frac{1}{t} k$

$$L = \int_1^e \sqrt{4t^2 + 4 + \frac{1}{t^2}}\, dt = \int_1^e \sqrt{\frac{4t^4 + 4t^2 + 1}{t^2}}\, dt$$

$$= \int_1^e \sqrt{\frac{(2t^2+1)^2}{t^2}}\, dt \quad \text{بس} \int_1^e \frac{2t^2+1}{t}\, dt = \int_1^e \left(2t + \frac{1}{t}\right)$$

حاصل التكامل

$$= [t^2 + \ln[t]]_{-1}^{e} = e^2 + 1 - 1 - 0 = \boxed{e^2}$$

* Speed on a smooth curve...

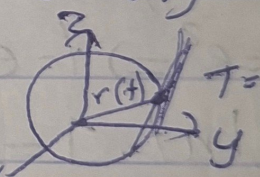
Since $s(t) = \int_{t_0}^{t} |v(\tau)| \, d\tau$ we have By the fundmental therem of calculas

$\frac{ds}{dt} = \underbrace{|v(t)|}_{=\text{speed}} > 0 \Rightarrow s(t)$ is an increasing fn

* for a smooth curve $\frac{ds}{dt} > 0$ because $|v|$ is never zero $\Rightarrow$ $s(t)$ is an increasing fnof t

اتجاه الحركة هو عبارة عن اتجاه الحركة

* Unit tangent vector T :-



we Know the velocity $V = \frac{dr}{dt}$ is a tangent to the curve So the unit tangent vector

$T = \frac{V}{|v|}$ since $\frac{ds}{dt} > 0$ we have $\frac{dt}{ds} > 0$

$$\Rightarrow \frac{dt}{ds} = \frac{1}{\frac{ds}{dt}} = \frac{1}{|v|}$$

So $\frac{dr}{ds} = \frac{dr}{dt} \frac{dt}{ds} = V\left(\frac{1}{|v|}\right) = \frac{V}{|v|} = T$

$\left|\frac{dr}{ds}\right| = \left|\frac{v}{|v|}\right| = \frac{|v|}{|v|} \Rightarrow \frac{dr}{ds}$ is a unit vector

tangent

Defs. The unit tangent vector of a smooth curve $r(t)$ is $T = \frac{dr}{ds} = \frac{v}{|v|}$

Example let $r(t) = \underbrace{(t\sin t + \cos t)}_{x} i + \underbrace{(t\cos t - \sin t)}_{y} j$, $t > 0$

(a) find the speed (b) find the unit tangent

(c) find the length of the curve for $\sqrt{2} \le t \le$

solution: $\frac{dx}{dt} = (\sin t + t \times \cos t - \sin t) = t\cos t$

$\frac{dy}{dt} = (\cos t - t \times \sin t - \cos t) = -t\sin t$

$v(t) = t\cos t\, i - t\sin t\, j$

speed $= |v(t)| = \sqrt{t^2\cos^2 t + t^2\sin^2 t} = \sqrt{t^2} = t$

(b) unit tangent vector $= \frac{v}{|v|} = \frac{t\cos t\, i - t\sin t\, j}{t}$

$T = \cos t\, i - \sin t\, j$

(c) length $= L = \int_{\sqrt{2}}^{2} t\, dt$

$\frac{1}{2}t^2\Big]_{\sqrt{2}}^{2} = \frac{1}{2}[4-2] = \frac{1}{2}(2) = 1$

التاريخ : / /

الدرس هو المتجه العمودي على المماس لهذا المنحنى

المتجه العمودي على المنحنى

13.4 Curvature and Normal vectors of a curve

التقوس أو الانحناء للمنحنى

Def:-

If T is the unit tangent vector of a smooth curve, the curvature fn. of a plane curve $K = \left|\frac{dT}{ds}\right|$ if a smooth curve $r(t)$ is given interms of the parameter t rather than the arc length s then $K = \left|\frac{dT}{ds}\right| = \left|\frac{dT}{dt}\,\frac{dt}{ds}\right| = \frac{\left|\frac{dT}{dt}\right|}{\left|\frac{ds}{dt}\right|} = \frac{\left|\frac{dT}{dt}\right|}{|v|}$

$\left|\frac{ds}{dt}\right|$ تسلسل متذكره

vector Kappa = scalar

ويعتبر scalar

Notes:-

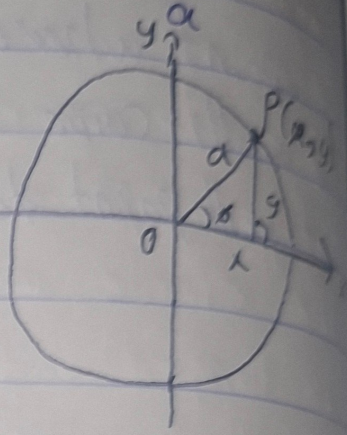
Since T has a constant length, we have $\frac{dT}{ds} \cdot T = 0$ that is $\frac{dT}{ds}$ and T are orthogonal

Example Show the curvature of a circle with center at the origin and radius is $\frac{1}{a}$

Solution

$x = a\cos\phi \qquad y = a\sin\phi$

the parametric eqns of the circle



$r(\phi) = a\cos\phi\, i + a\sin\phi\, j$

$v(\phi) = \frac{dr}{dt} = -a\sin\phi\, i + a\cos\phi\, j$

$|v| = \sqrt{a^2\sin^2\phi + a^2\cos^2\phi} = \sqrt{a^2(\sin^2\phi + \cos^2\phi)} =$

$T = \frac{V}{|v|} = \frac{-a\sin\phi\, i + a\cos\phi\, j}{a} = -\sin\phi\, i + \cos$

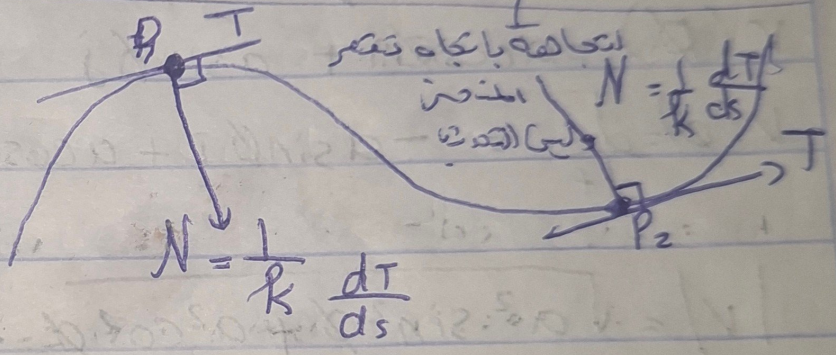
$\frac{dT}{d\phi} = -\cos\phi\, i - \sin\phi\, j \Rightarrow \left|\frac{dT}{d\phi}\right| = \sqrt{\cos^2\phi +$

$K = \frac{\left|\frac{dT}{d\phi}\right|}{|v|} = \boxed{\frac{1}{a}}$

* The principle unit normal vector

Def:

At a point where $K \neq 0$, the principal unit vector of a smooth curve in the plane $N = \frac{1}{K}\frac{dT}{ds}$ that will point toward to concave side on the curve



Remark:

we can find the vector N without finding K and s as follows $N = \frac{1}{K}\frac{dT}{ds} = \frac{\frac{dT}{ds}}{\left|\frac{dT}{ds}\right|} = \frac{\frac{dT}{dt}\frac{dt}{ds}}{\left|\frac{dT}{dt}\frac{dt}{ds}\right|}$

$$N = \frac{\frac{dT}{dt}}{\left|\frac{dT}{dt}\right|}$$

Example: find T, N and K for the plan curve

$r(t) = (\ln \sec t)\, i + t\, j \Rightarrow -\frac{\pi}{2} < t < \frac{\pi}{2}$

Solution:

$$v = \frac{dr}{dt} = \frac{1}{\sec t} \sec t \tan t\, i + 1\, j$$

$$\boxed{v = \tan t\, i + j}$$

$$|v| = \sqrt{\tan^2 t + 1} = \sqrt{\sec^2 t} = \sec t$$

ملاحظة: قيمة الـ sec دايما الأول والتاني موجب

$$T = \frac{v}{|v|} = \frac{\tan t\, i + j}{\sec t} = \sin t\, i + \cos t\, j$$

$$\frac{dT}{dt} = \cos t\, i - \sin t\, j$$

$$\left|\frac{dT}{dt}\right| = \sqrt{\cos^2 t + \sin^2 t} = 1$$

$$\boxed{N = \cos t\, i - \sin t\, j}$$

$$K = \frac{\left|\frac{dT}{dt}\right|}{|v|} = \frac{1}{\sec t} \Rightarrow \boxed{K = \cos t}$$

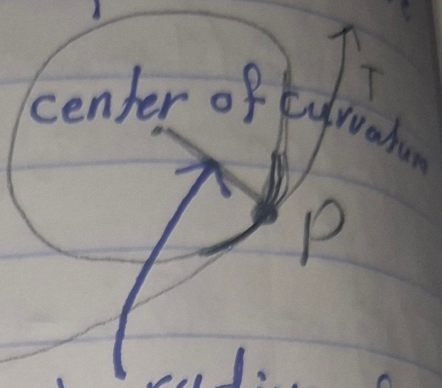
* Circle of curvature for plane Curve

Defs.

① The circle of curvature (osculating circle) at a point p an a plane curve where $K \neq 0$ is the circle in the plane of the curve that
(a) is tangent to The curve at p
(b) is the same curvature as the curve at p
(c) lies toward the concave inner side of the curve

② The ~~ex~~ vectors of curvature of the curve at p is the circle of curvature and this radius is $\rho = \frac{1}{K}$

③ the center of curveture of the curve at p is the center its curvature of the circle of curvature



$\rho = \frac{1}{k}$ radius of cur

Examples: find the graph the asculating circle of the curve $r(t) = ti + \sin t j$ at t

Solution: يتم حل بالصورة البارامترية بدي أحوله للصورة الكارتيزية

$x = t$, $y = \sin x \Rightarrow$

$$\boxed{y = \sin x}$$

$$v = \frac{dr}{dt} = i + \cos t\, j$$

$$|v| = \sqrt{1+\cos^2 t} \Rightarrow |v(\tfrac{\pi}{2})| = \sqrt{1+\cos^2 \tfrac{\pi}{2}} = \sqrt{}$$

$$T = \frac{v}{|v|} = \frac{1}{\sqrt{1+\cos^2 t}} = \frac{1}{\sqrt{1+\cos^2 t}}\, i + \frac{\cos t}{\sqrt{1+\cos^2 t}}\, j$$

$$T = (1+\cos t)^{\frac{1}{2}}\, i + \cos t\,(1+\cos^2 t)^{-\frac{1}{2}}\, j$$

$$\frac{dT}{dt} = \frac{-1}{2}(1+\cos^2 t)^{\frac{-3}{2}} \times (2\cos t \sin t)\, i + \text{بدي أكمل المشتقة}$$

$$+\left(-\sin t\left(1+\cos^2 t\right)^{-\frac{1}{2}}+-\frac{1}{2}\cos t\left(1+\cos^2 t\right)^{-\frac{3}{2}}\right)i$$
$$(-2\cos t \sin t)\,j$$

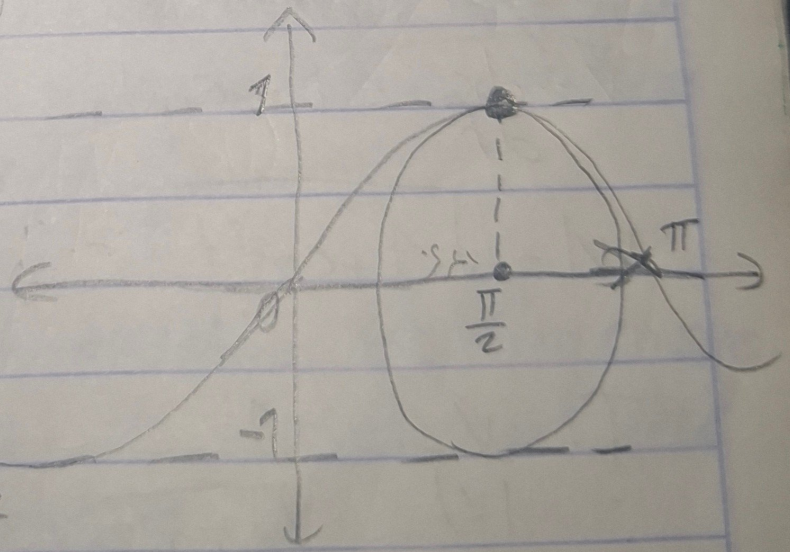
$$\frac{dT}{dt}\left(\frac{\pi}{2}\right) = 0i - j = \boxed{-j} \#$$

$$\left|\frac{dT}{dt}\right| = 1$$

$$K\left(\frac{\pi}{2}\right) = \left|\frac{\frac{dT}{dt}\left(\frac{\pi}{2}\right)}{v\left(\frac{\pi}{2}\right)}\right| = \frac{1}{1} = 1$$

The radius of the circle of curvature

$$\rho = \frac{1}{K} = \frac{1}{1} = 1$$



$$(x-y)^2 + (y-K)^2 = a^2$$

eqn. of the circle of curvature

$$\boxed{\left(x-\frac{\pi}{2}\right)^2 + y^2 = 1} \#$$

13.5 Tangential and Normal Components of Acceleration

المركبات المماسية والعمودية للعجلة

⇒ the velocity of a particle with position r is $v = \frac{dr}{dt} = \frac{dr}{ds}\frac{ds}{dt} = \frac{ds}{dt}T$

T and $\frac{dr}{ds}$ are parallel

$$a = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{ds}{dt}T\right)$$

$$= \frac{d^2s}{dt^2}T + \frac{ds}{dt}\frac{dT}{dt} \Rightarrow \frac{ds^2}{dt^2}T + \frac{ds}{dt}\left(\frac{dT}{ds}\frac{ds}{dt}\right)$$

$$= \frac{ds^2}{dt^2}T + \frac{ds}{dt}(\kappa N)\frac{ds}{dt}$$

scalar scalar

$$\Rightarrow a = \frac{ds^2}{dt^2}T + \kappa\left(\frac{ds}{dt}\right)^2 N$$

$a_T = \frac{ds^2}{dt^2} = \frac{d}{dt}\left(\frac{ds}{dt}\right) = \frac{d}{dt}(|v|)$ is the scalar component of acceleration

tangential

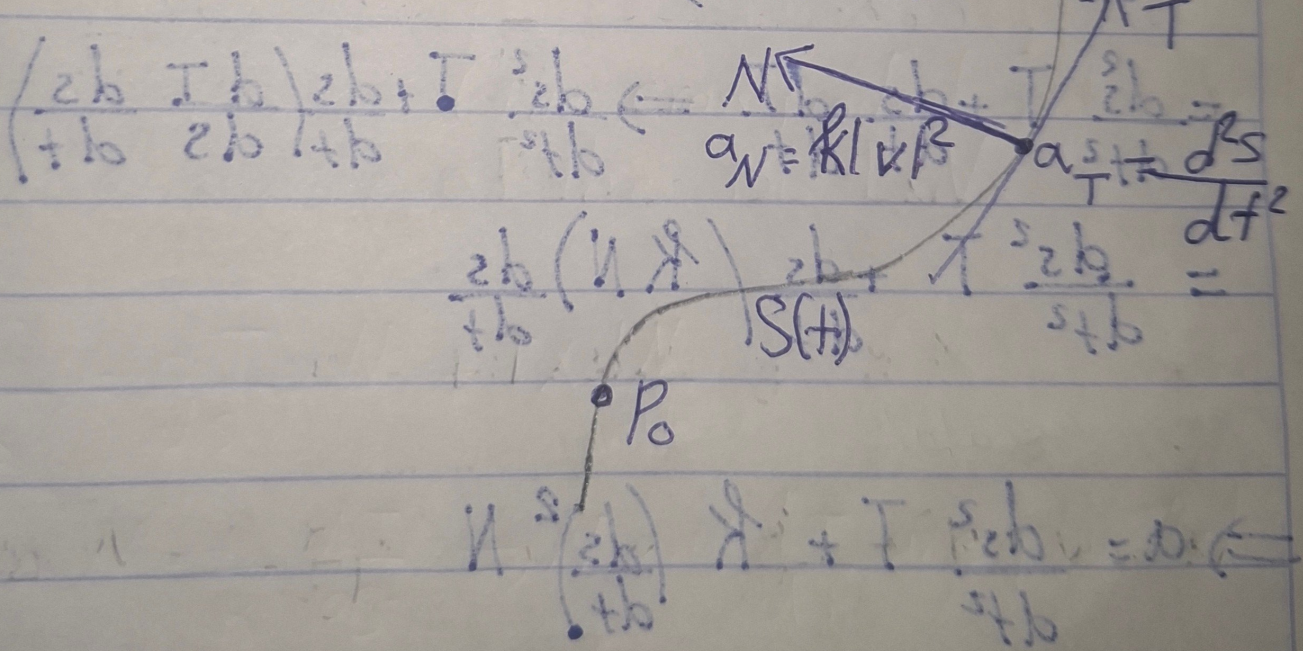
معدل تغير مقدار السرعة ← معدل التغير

unit tangent / المركبة المماسية للعجلة باتجاه

$a_{(N)} = k\left(\frac{ds}{dt}\right)^2 = k|v|^2$ is the scalar normal compond of acceleration

Notes $a \cdot a = |a|^2$

$a_T^2 + a_N^2 = a \cdot a = |a|^2 \Rightarrow a_N = \sqrt{|a|^2 - a_T^2}$



Example let $r(t) = (t+1)i + 2tj + t^2k$ write the acceleration a in the form $a = a_T T + a_N N$ at $t=1$ and N without finding T

solution:

$$v(t) = \frac{dr}{dt} = i + 2j + 2tk$$

$$|v| = \sqrt{1+4+4t^2} = \sqrt{5+4t^2}$$

$$a_T = \frac{d}{dt}(|v|) = \frac{1}{2\sqrt{5+4t^2}} \times 8t = \frac{4t}{\sqrt{5+4t^2}}$$

$$a_T(1) = \frac{4}{\sqrt{5+4}} = \frac{4}{\sqrt{9}} = \boxed{\frac{4}{3}}$$

$$a(t) = 2k \Rightarrow a \cdot a = |a|^2 = 4$$

$$a_N = \sqrt{|a|^2 - a_T^2} = \sqrt{4 - \left(\frac{4}{3}\right)^2} = \sqrt{4 - \frac{16}{9}} =$$

$$= \boxed{\frac{2\sqrt{5}}{3}}$$

$$\boxed{a_t = \frac{4}{3}T + \frac{2\sqrt{5}}{3}N}$$

unit vector → unit vector

Defs.

The binormal vector of a space curve is $B = T \times N$, is a unit vector orthogonal to both T and N

(is a unit vector)

unit vector × unit vector = unit vector

$$u = \frac{1}{\sqrt{3}}i + \frac{1}{\sqrt{3}}j + \frac{1}{\sqrt{3}}K \qquad u \times i = \begin{vmatrix} i & j & K \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 1 & 0 & 0 \end{vmatrix}$$

$$1\left(\frac{1}{\sqrt{3}}j - \frac{1}{\sqrt{3}}K\right)$$

(The cross product not commutative.)

The derivative of the $B = T \times N$

$$\frac{dB}{ds} = \frac{dT}{ds} \times N + T \times \frac{dN}{ds}$$

(unit vector)

since $N = \frac{1}{K}\frac{dT}{ds}$ (scalar, vector) $\Rightarrow N$ and $\frac{dT}{dt}$ are parallel

$$\Rightarrow N \times \frac{dT}{ds} = 0 \qquad \Rightarrow \frac{dB}{ds} = T \times \frac{dN}{ds}$$

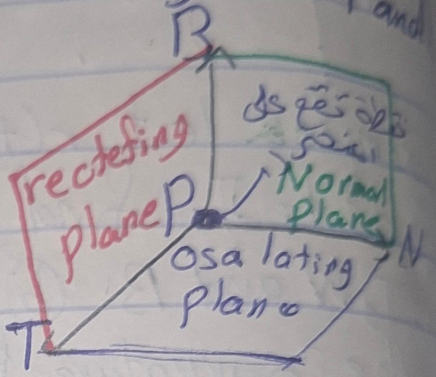
$\Rightarrow \frac{dB}{ds}$ is orthogonal to both T and $\frac{dN}{ds}$ ... (1)

since B is a constant vector we have
$B \cdot \frac{dB}{ds} = 0 \Rightarrow \frac{dB}{ds}$ is orthogonal to B.

From ① and ② $\frac{dB}{ds}$ is orthogonal T and

$\Rightarrow \frac{dB}{ds}$ is parallel to the

Normal vector (N)



$$\Rightarrow \frac{dB}{ds} = -\tau N \Rightarrow \frac{dB}{ds} \cdot N = -\tau (N \cdot N)$$

$$\boxed{\tau = -\frac{dB}{ds} \cdot N}$$

Def: the torsion fn of a smooth curve is $\tau = -\frac{dB}{ds} \cdot N$, where τ is the rate at which the osculating plans turns about T

Remark: from advaned calculas we have

① the curvature $k = \left|\frac{dT}{ds}\right| = \frac{|v \times a|}{|v|^3}$

② the tourion $\tau = -\frac{dB}{ds} = \frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{|v \times a|^2}$

$(v \times a \neq 0)$

Example: let $r(t) = 3\sin t\, i + 3\cos t\, j + 4t\, K$

at $t = \frac{\pi}{2}$ find

(a) T, N and B (b) the eqn the normal plane

(c) the curvature k and the torion τ

solution: $v = \frac{dr}{dt} = 3\cos t\, i - 3\sin t\, j + 4K$

$|v| = \sqrt{9\cos^2 t + 9\sin^2 t + 16} = \sqrt{9+16} = \boxed{5}$

$T = \frac{v}{|v|} = \frac{3}{5}\cos t\, i - \frac{3}{5}\sin t\, j + \frac{4}{5}K$

$\boxed{T\left(\frac{\pi}{2}\right) = \frac{-3}{5}j + \frac{4}{5}K}$ $\quad v\left(\frac{\pi}{2}\right) = -3j + 4K$

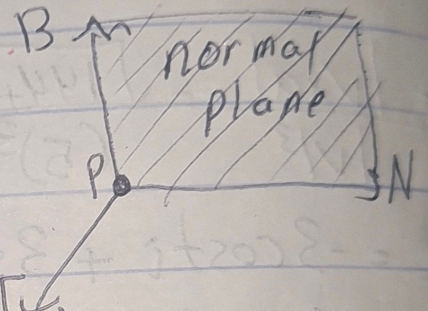
$\frac{dT}{dt} = \frac{-3}{5}\sin t\, i - \frac{3}{5}\cos t\, j$

$\left|\frac{dT}{dt}\right| = \sqrt{\frac{9}{25}\underbrace{\sin^2 t}_{1} + \frac{9}{25}\cos^2 t} = \boxed{\frac{3}{5}}$

$$N = \frac{\frac{dT}{dt}}{\left|\frac{dT}{dt}\right|} = -\sin t\, i - \cos t\, j \Rightarrow N\left(\frac{\pi}{2}\right) = -i$$

$$* B = T \times N = \begin{vmatrix} i & j & K \\ 0 & -\frac{3}{5} & \frac{4}{5} \\ -1 & 0 & 0 \end{vmatrix} = -1\left(\frac{4}{5}j + \frac{3}{5}K\right)$$

$$= \boxed{-\frac{4}{5}j - \frac{3}{5}K}$$



$$(x, y, z) = \left(3\sin\frac{\pi}{2}, 3\cos\frac{\pi}{2}, 4\,\frac{\pi}{2}\right)$$

$$P = (3, 0, 2\pi)$$

$T\left(\frac{\pi}{2}\right)$ is the normal vector to the normal plane

$$0 - \frac{3}{5}y + \frac{4}{5}z = 0\left(\frac{3}{5}\right) + 0\left(\frac{-3}{5}\right) + \frac{4}{5}(2\pi)$$

$$= -\frac{3}{5}y + \frac{4}{5}z = \frac{8\pi}{5} \Rightarrow \boxed{-3y + 4z = 8\pi}$$ (b)

equation of the normal plane

$$a\left(\frac{\pi}{2}\right) = -3i$$

$$|v \times a| = \begin{vmatrix} i & j & K \\ 0 & -3 & 4 \\ -3 & 0 & 0 \end{vmatrix} = -3(4j + 3K) = -12j - 4K$$

$$K = \frac{|v \times a|}{|v|^3} = \frac{\sqrt{144+81}}{(5)^3} = \frac{15}{(5)^3} = \boxed{\frac{3}{25}}$$

$$\frac{da}{dt} = -3\cos t\, i + 3\sin t\, j \Rightarrow \frac{da}{dt}\left(\frac{\pi}{2}\right) = \boxed{3j}$$

$$T = \frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{|v \times a|^2} = \frac{\begin{vmatrix} 0 & -3 & 4 \\ -3 & 0 & 0 \\ 0 & 3 & 0 \end{vmatrix}}{(15)^2}$$

$$= \frac{4(-4)}{(15)^2} = \boxed{\frac{-4}{25}}$$